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Abstract

Full Text

MATHEMATICS

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SEVERAL REMARKS ON FLOWS OF LINEAR ELEMENTS AND FRAMES

(Presented by Academician A. N. Kolmogorov, 9 XII 1960)

Many problems of mechanics lead, as is well known, to **geodesic flows** (see [1–4]). In the present note more general dynamical systems in the spaces of linear elements and frames of a Riemannian manifold are defined—**isotropic flows**. These include the flow associated with curves of constant geodesic curvature and with the motion of a charged point on a smooth surface in the presence of a magnetic field.

§ 1. Let M be an n -dimensional Riemannian manifold. A k -frame, or k -hedron, on M is a pair $\omega = (x, \xi^k)$, where $x \in M$ is the carrier, and ξ^k is an ordered set (ξ_1, \dots, ξ_k) of pairwise orthogonal unit vectors of the tangent space to M at the point x . Frames with a common carrier form a homogeneous space Ξ_k , and all frames on M form the space Ω_k . We define the volume element in the space Ω_k by the formula $d\Omega = dM d\Xi$, where dM is the volume element on M , and $d\Xi$ determines the invariant measure in the space Ξ_k .

By a **flow of k -hedra** we shall mean a one-parameter group S^t of transformations of the space Ω_k : $\omega \rightarrow S^t\omega$. The **trajectories** of the flow will be the lines Γ on M formed by the carriers $x(t)$ of the elements $S^t\omega$. By a **tangent flow** we shall mean such a flow of n -hedra for which the frame $S^t\omega$ is the accompanying frame of the trajectory $x(t)$. Denote by v the speed of motion of $x(t)$ along Γ , and by k_1, \dots, k_{n-1} the curvatures of Γ . Clearly, a tangent flow is determined by specifying the functions $v(\omega), k_1(\omega), \dots, k_{n-1}(\omega)$ on Ω_n .

Definition. An **isotropic flow** is a tangent flow for which the speed v is constant, while the curvatures $k_1(x), \dots, k_{n-1}(x)$ depend only on the carrier x and do not depend on the directions of the vectors ξ_1, \dots, ξ_n of the frame ω .

In particular, the geodesic flow is isotropic: $v \equiv 1, k_i \equiv 0$.

§ 2. It turns out that an isotropic flow is a dynamical system with an integral invariant. As is known, the measure $d\Omega$ is an invariant measure of the geodesic flow.

Theorem 1. The transformations S^t of any isotropic flow preserve the measure $d\Omega$.

The proof is based on the fact that the infinitesimal transformation S^{dt} consists of an infinitesimal transformation of the geodesic flow and an infinitesimal rotation.

For $n = 2$, an isotropic flow is determined by the geodesic curvature $k(x)$ of the trajectory passing through the point x of the surface M . Such a flow is isomorphic to motion at fixed energy in a dynamical system with Lagrangian function $L_2 + L_1$, containing terms quadratic and linear in the velocities. In this case Theorem 1 follows from Liouville's theorem. Eliminating the cyclic coordinate, one can easily

to investigate the course of curves of constant curvature on surfaces of revolution. These curves were studied by Minding ⁽⁵⁾ and Darboux ⁽⁶⁾.

§ 3. Also readily amenable to study are **cyclic** flows, in which k_1, \dots, k_{n-1} are constant, on manifolds of constant curvature K . Let us consider the Lobachevskii plane ($n = 2, K = -1$). A standard device makes it possible to pass to any surface of constant negative curvature ⁽²⁾.

Cycles of curvature k on the Lobachevskii plane are divided into three classes: cycles proper ($k^2 > 1$), horicycles ($k^2 = 1$), and hypercycles ($k^2 < 1$). In the Poincaré model these are, respectively, circles not intersecting the absolute, tangent to it, and intersecting the absolute at an angle α ; $\cos \alpha = k$.

Theorem 2. *Every hypercyclic ($k^2 + K < 0$) flow on a surface of constant negative curvature K is isomorphic to a geodesic flow.*

The proof is based on the fact that a hypercycle of curvature k is an equidistant curve Γ_τ , i.e. it is separated from some geodesic Γ by a constant distance τ (where $k = -\sqrt{-K} \operatorname{th} \sqrt{-K}\tau$). With the equidistants Γ_τ on any surface one can associate a tangent flow isomorphic to the geodesic flow. On a surface of constant curvature the flow of equidistants Γ_τ is cyclic.

§ 4. Analogous considerations in Lobachevskii space make it possible to prove the following:

Theorem 3. *Every cyclic flow on an n -dimensional manifold of constant negative curvature -1 belongs to one of the following three types:**

1. *The flow is isomorphic to the generalized geodesic flow ($k_1 = 0$).*
2. *The flow is isomorphic to the generalized horicyclic flow ($k_1 = 1, k_2 = 0$).*
3. *The flow is isomorphic to one for which the carrier $S^t\omega$ is fixed.*

The usual geodesic flow belongs to flows of the first type (for it $\nu = 1, k_1 = k_2 = \dots = k_{n-1} = 0$); all these flows are very similar to it (see Theorem 5). Among flows of the second type the ordinary horicyclic flow plays the same role ($\nu = k_1 = 1; k_2 = \dots = k_{n-1} = 0$). Flows of the third type are not ergodic; the ergodic components are tori of dimension r ; generally speaking, $r = [n/2]$; on each torus the flow has a discrete spectrum with r generators.

The question of which type a given flow belongs to is decided as follows:

Theorem 4. *In a $2r$ -dimensional space of curvature -1 , a cyclic flow belongs to type 1, 2, or 3 according as k_1^2 is less than, equal to, or greater than*

$$\chi^2 = 1 + \frac{k_2^2}{k_3^2} + \frac{k_2^2 k_4^2}{k_3^2 k_5^2} + \dots + \frac{k_2^2 k_4^2 \dots k_{2r-2}^2}{k_3^2 k_5^2 \dots k_{2r-1}^2}.$$

In a $(2r + 1)$ -dimensional space all flows with $k_{2r} \neq 0$ are of type 1, while for $k_{2r} = 0$ the flow is of type 1, 2, or 3 according as k_1^2 is less than, equal to, or greater than χ^2 .

§ 5. The methods of recent works of Ya. G. Sinai ^(4,7), devoted to the ordinary geodesic flow, are applicable also to the generalized one.

Theorem 5. *A flow S^t of type 1 on a compact manifold of constant negative curvature is a K -system ^(4,7). The horospherical flow ^(4,7) is conjugate to S^t .*

One can calculate the entropy ⁽⁸⁾ of the flow S^t . Denote by $h(k_1, \dots, k_{n-1})$ the entropy per unit time of the cyclic flow with velocity 1 and

* In each two-dimensional direction.

curvatures k_1, \dots, k_{n-1} on a manifold of constant curvature -1 . Then

$$h(k_1, \dots, k_{n-1}) = h(0)v,$$

where $h(0) = h(0, \dots, 0)$ is the entropy of the geodesic flow, computed in ⁽⁴⁾, and $v(k_1, k_2, \dots, k_{n-1})$ is the speed of motion in the generalized geodesic flow isomorphic to S^t .

In particular, for $n = 2$ we have

$$h(k) = h(0) \sin \alpha = h(0) \sqrt{1 - k^2},$$

and for $n = 3$ the number $v^2 = x$ is the positive root of the equation

$$x^2 + (k_1^2 + k_2^2 - 1)x + k_2^2 = 0.$$

In the case of a surface ($n = 2$), when $k^2 \rightarrow 1$, then $h \rightarrow 0$. This suggests that the entropy of the horocyclic flow $h(1)$ is zero. The latter was proved by B. M. Gurevich ⁽⁹⁾. Probably the entropy of any flow of type 2 is zero.

§ 6. The spectra of the geodesic and horocyclic flows were first found ^(3,10) on the basis of an algebraic construction of I. M. Gelfand and S. V. Fomin; this construction depends on a certain group G and its subgroups: a compact subgroup K , a discrete subgroup D , and a one-parameter subgroup g^t .

Theorem 6. *If the group G is the group of motions of n -dimensional Lobachevsky space and K is the rotation group of an $(n - k)$ -dimensional*

Euclidean space, then the corresponding dynamical system is isomorphic to one of the cyclic flows of k -frames on a manifold of constant negative curvature; conversely, all such flows can be obtained in this way.

The proof is based on the connection between curves of constant curvature and one-parameter groups of motions; conjugate groups correspond to isomorphic systems.

§ 7. An interesting topological characteristic of a dynamical system is the group of rotation numbers ^(11, 12). I. M. Gelfand and I. I. Pyatetskii-Shapiro computed the rotation numbers of the geodesic and horocyclic flows on a surface of constant negative curvature ⁽¹³⁾.

Theorem 7. *If an isotropic flow on a Riemannian manifold distinct from the two-dimensional torus and from the Klein bottle is ergodic, then all rotation numbers are equal to zero and the flow has no nonconstant continuous eigenfunctions.*

The proof is based on the fact that the trajectory of an ergodic isotropic flow passes equally often through each point in every direction.

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