AN EXISTENCE THEOREM FOR RATIONAL FUNCTIONS ON RIEMANN SURFACES

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Abstract

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MATHEMATICS

V. G. MIKHAL' CHUK

AN EXISTENCE THEOREM FOR RATIONAL FUNCTIONS ON RIEMANN SURFACES

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1. In order that on a closed Riemann surface R of genus g there exist a rational analytic function with poles of the first order at prescribed points p_μ ($\mu = 1, 2, \ldots, n$), respectively with principal parts a_μ / z, it is necessary and sufficient that the equality

$$\sum_{\mu=1}^{n} a_\mu \varphi(p_\mu) = 0, \tag{1}$$

hold, where $\varphi(p)$ is an arbitrary covariant of the first kind on R.

This assertion may be derived from the following relation (see ([1]), p. 204):

$$\int_J f(z)\, dw = -\sum_{\nu=1}^{g} A'_\nu B_\nu = 2\pi i \sum_{\mu=1}^{n} a_\mu \varphi(p_\mu), \tag{2}$$

where J is a system of canonical cuts $K_1, K_2, \ldots, K_{2g}$ of the surface R; dw is an Abelian differential of the first kind with periods A'_ν, B'_ν along $K_{2\nu-1}$, respectively $K_{2\nu}$; $df(z)$ is a complexly normalized Abelian differential of the second kind with principal parts $-a_\mu / z^2$ at the points p_μ ($\mu = 1, 2, \ldots, n$) and with periods A_ν, B_ν along $K_{2\nu-1}$, respectively $K_{2\nu}$, with $A_\nu = 0$ (the condition of complex normalization). Here z denotes a local parameter of the surface R.

The aim of what follows is to generalize the indicated assertion to the case of quasianalytic functions.

2. By a quasianalytic function $f(z)$ we mean a continuously differentiable solution of an equation of the form

$$w_{\bar{z}} = B(z)\overline{w_z}, \qquad |B(z)| < 1. \tag{3}$$

Let $df(z)$ be a complexly normalized quasianalytic differential (it can be made such analogously to the case of an analytic differential), belonging to equation

(3), with poles of the second order at the points p_μ $(\mu = 1, 2, \ldots, n)$ and, respectively, with principal parts $\alpha_\mu dZ_{x_\mu}(z, z_\mu) + i\beta_\mu dZ^1_{x_\mu}(z, z_\mu)$, where α_μ, β_μ are real coefficients (see ([2]), p. 30). Further, let $dw = du + i\,dv$ be an arbitrary quasianalytic differential of the first kind, and let dw_k $(k = 1, 2, \ldots, 2g)$ be a basis of differentials of the first kind belonging to the equation

$$w'_{\bar z} = -\overline{B(z)}\,\overline{w'_z}. \tag{4}$$

Let us note that the basis of quasianalytic differentials is a $2g$-dimensional real space, i.e., it has $2g$ linearly independent components with respect to real coefficients and is not reduced to a g-dimensional complex space, since a differential belonging to equation (4), multiplied by a complex number, will not belong to this equation.

Let us now consider the integral

$$H = \int_J f(z)\,dw + \overline{f(z)}\,d\overline{w}.$$

Preserving the notation for the periods of the preceding paragraph for the differentials $df(z)$ and dw, we obtain

$$H = -2\,\mathrm{Re}\sum_{\nu=1}^{g} A'_\nu B_\nu. \tag{5}$$

The integrand of the integral H is a complete differential; therefore its residues can be computed (see (2), p. 30)

$$H = -\int_J w\,df + \overline{w}\,d\overline{f} = 4\pi\sum_{\mu=1}^{n}\left[\alpha_\mu\frac{\partial v(p_\mu)}{\partial x} + \beta_\mu\frac{\partial u(p_\mu)}{\partial x}\right]. \tag{6}$$

Theorem 1. *In order that on the closed Riemann surface R there exist a single-valued quasianalytic function of equation (3) with poles of the first order at the points p_μ $(\mu = 1, 2, \ldots, n)$, respectively with principal parts $\alpha_\mu Z_{x_\mu} + \mathrm{i}\beta_\mu Z^1_{x_\mu}$ $(\mu = 1, 2, \ldots, n)$, it is necessary and sufficient that the inequality*

$$\sum_{\mu=1}^{n}\left[\alpha_\mu\frac{\partial v(p_\mu)}{\partial x} + \beta_\mu\frac{\partial u(p_\mu)}{\partial x}\right] = 0, \tag{7}$$

hold, where

$$dw = \frac{\partial(u+iv)}{\partial x}\,dx + \frac{\partial(u+iv)}{\partial y}\,dy$$

is an arbitrary quasianalytic differential of the first kind, belonging to equation (4) on the surface R.

Proof. Necessity follows from comparing formulas (5) and (6).

Sufficiency. Let $df(z)$ be a complex-normalized quasianalytic differential of the second kind belonging to equation (3), with poles at the points p_μ ($\mu = 1, 2, \ldots, n$) and with principal parts respectively equal to $\alpha_\mu dZ_{x_\mu} + \mathrm{i}\beta_\mu dZ^1_{x_\mu}$. Suppose that condition (7) is satisfied. Then, from comparing formulas (5) and (6), we obtain

$$\operatorname{Re}\sum_{\nu=1}^{g} A'_\nu B_\nu = 0. \tag{8}$$

Condition (8) is true for any quasianalytic differential of the first kind belonging to equation (4); hence it holds also for the basis differentials dw_k ($k = 1, 2, \ldots, 2g$), belonging to the same equation, i.e.,

$$\operatorname{Re}\sum_{\nu=1}^{g} B_\nu A_{k\nu} = 0 \qquad (k = 1, 2, \ldots, 2g),$$

where A_{kv} is the period of the differentials dw_k with respect to the paths K_{2v-1}. If

$$B_v = b_v^{(1)} + ib_v^{(2)}, \qquad A_{kv} = a_{kv}^{(1)} + ia_{kv}^{(2)},$$

then the last system can be written in the form

$$\sum_{v=1}^{g} \left[a_v^{(1)} b_{kv}^{(1)} - b_v^{(2)} a_{kv}^{(2)}\right] = 0 \qquad (k = 1, 2, \ldots, 2g). \tag{9}$$

The determinant of the system (9) can differ only in sign from the determinant of the system

$$a_k^{(1)} = \sum_{v=1}^{2g} c_v a_{vk}^{(1)},$$

$$a_k^{(2)} = \sum_{v=1}^{2g} c_v a_{vk}^{(2)} \qquad (k = 1, 2, \ldots, g), \tag{10}$$

where c_v are real coefficients determining a certain differential

$$dw = \sum_{v=1}^{2g} c_v dw_v \tag{11}$$

with periods $a_k^{(1)} + ia_k^{(2)}$ with respect to the paths K_{2k-1}. But the determinant of the system (10) is different from zero by virtue of the uniqueness of the determination of the differential dw by arbitrarily prescribed periods $a_k^{(1)} + ia_k^{(2)}$. Consequently, the determinant of the system (9) is different from zero, and therefore the system has only the trivial solution $B_v = 0$. Hence the function $f(z)$ is single-valued on R.

3. The assertion given in no. 1 concerning the existence on the surface R of a rational analytic function with poles of the first order can be generalized also to the case of poles of higher orders. If the function $f(z)$ has at the points p_μ $(\mu = 1, 2, \dots, n)$ poles of order m_μ with principal parts

$$\frac{a_{m_\mu}^{(\mu)}}{z^{m_\mu}} + \frac{a_{m_\mu-1}^{(\mu)}}{z^{m_\mu-1}} + \dots + \frac{a_1^{(\mu)}}{z},$$

then instead of condition (1) we shall have

$$\sum_{\mu=1}^{n} \left[\frac{a_{m_\mu}^{(\mu)}}{(m_\mu - 1)!} \frac{d^{m_\mu-1}}{dz^{m_\mu-1}} \varphi(p_\mu) + \dots + \frac{a_2^{(\mu)}}{1!} \frac{d}{dz} \varphi(p_\mu) + a_1^{(\mu)} \varphi(p_\mu) \right] = 0. \tag{12}$$

This is not difficult to verify by computing the residues of the expression $f(z)\,dw$. It should be noted that the quantities $a_{m_\mu}^{(\mu)}, a_{m_\mu-1}^{(\mu)}, \dots, a_1^{(\mu)}$ depend in a definite way on the choice of the local parameter; however, each bracketed expression in relation (12) is invariant with respect to the choice of the local parameter.

An analogous generalization also holds for quasianalytic functions, namely, the following theorem holds:

Theorem 2. *In order that on a closed Riemann surface R there exist a rational quasianalytic function satisfying*

···of (3) with poles at the points p_μ $(\mu = 1, 2, \dots, n)$ of order m_μ, and in them with principal parts

$$\sum_{j=1}^{m_\mu} \left[\alpha_j^{(\mu)} Z_{x_\mu^j}(z, z_\mu) + i\beta_j^{(\mu)} Z_{y_\mu^j}(z, z_\mu) \right],$$

it is necessary and sufficient that the equality

$$\sum_{\mu=1}^{n}\left\{\sum_{j=1}^{m_\mu}\left[\frac{\alpha_j^{(\mu)}}{(j-1)!}\frac{\partial^{j-1}v(p_\mu)}{\partial x^{j-1}}+\frac{\beta_j^{(\mu)}}{(j-1)!}\frac{\partial^{j-1}u(p_\mu)}{\partial x^{j-1}}\right]\right\}=0, \tag{13}$$

where $dw = du + i\,dv$ is an arbitrary quasianalytic differential of the first kind belonging to equation (4) on the surface R.

For the proof of Theorem 2 one should require the existence, in a neighborhood of the points p_μ, of derivatives of the functions $\dot{B}(z)$ (see equation (3)) of order $(m_\mu - 1)$.

4. Theorems 1 and 2 hold for solutions of a more general elliptic system of differential equations, namely

$$v_y = au_x + bu_y,$$

$$v_x = cu_x + du_y,$$

which in complex form may be written as follows:

$$w_{\bar z} = A_1(z)w_z + B_1(z)\bar{w}_{\bar z}, \tag{14}$$

where

$$A_1 = -\frac{p_1(p^2-1)}{(pp_1+1)(p+p_1)}e^{2i\theta}, \qquad B_1 = \frac{p(p_1^2-1)}{(pp_1+1)(p+p_1)}e^{2i\theta_1},$$

$p, \theta;\ p_1, \theta_1$ are the characteristics of the quasiconformal mapping $w = F(z)$ (see (2)). This follows from the following considerations. If on the Riemann surface R a solution of differential equation (14) is given, then by choosing a new local parameter $\zeta = \varphi(z)$, which in each parametric circle $|z| < 1$ satisfies the equation

$$\varphi_{\bar z} = A(z)\varphi_z, \qquad A = -\frac{p-1}{p+1}e^{2i\theta},$$

one passes to the function $w = f(\zeta)$, which is a solution of equation (3) with respect to the local parameter ζ on the surface R_1, obtained from R by changing the local parameters.

Perm State Pedagogical Institute

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Note: Figure translations are in progress. See original paper for figures.