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Abstract

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THE GEOMETRY OF TWO ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Differential equations as a subject of geometrical investigation have repeatedly attracted the attention of geometers. In particular, questions connected with the geometry of an ordinary differential equation of the second order were studied by Cartan ([1]) and Dubourdieu ([2]). A. M. Vasil' ev ([3]) indicated a way of constructing the most general geometric theory of differential equations, connected with the use of invariant differential-geometric methods of G. F. Laptev ([4]), generalized to infinite Lie groups.

The aim of the present work is to find properties of the totality of integral curves of two ordinary differential equations of the second order that are invariant with respect to arbitrary analytic transformations of the plane. The basic method of investigation is the canonization of the frame ([5]) attached to a manifold immersed in the representation space of an infinite Lie group. The canonization algorithm in this case does not differ from the usual one (see, for example, ([6])).

It should be noted (in what follows this will not be stipulated) that all constructions of the present work are of a strictly local character; all functions occurring in the work are assumed to be analytic.

Consider a two-dimensional plane S, on which acts the infinite group G of all point analytic transformations. For the purposes of the problem it will also be expedient to consider the representation of the group G on the set of elements of the form: a point plus two tangent directions issuing from this point. Let us prescribe on S a pair of ordinary differential equations of the second order (in symmetric notation):

$$y'' = f(x, y, y'), \tag{1}$$
$$x'' = \varphi(x, y, x'). \tag{2}$$

It turns out to be possible to attach to equations (1), (2), in a manner invariant with respect to transformations of the group G, a system of differential forms

$\omega^1,\ \omega^2,\ \omega_2^1,\ \omega_1^2,\ \omega_1^1,\ \omega_2^2,\ \omega_{21}^1,\ \omega_{12}^2$, determined up to an arbitrary analytic transformation of the parameters and satisfying the following structure equations:

$$\begin{aligned}
D\omega^1 &= [\omega^1\omega_1^1] + [\omega^2\omega_2^1],\\
D\omega^2 &= [\omega^1\omega_1^2] + [\omega^2\omega_2^2],\\
D\omega_2^1 &= [\omega_2^1, \omega_1^1 - \omega_2^2] + [\omega_{21}^1\omega^1],\\
D\omega_1^2 &= [\omega_1^2, \omega_2^2 - \omega_1^1] + [\omega_{12}^2\omega^2],\\
D\omega_1^1 &= [\omega_1^2\omega_2^1] + [\omega_{21}^1\omega^2] + 2[\omega_{12}^2\omega^2] + 3\alpha[\omega^1\omega_2^1],\\
D\omega_2^2 &= [\omega_2^1\omega_1^2] + [\omega_{12}^2\omega^1] + 2[\omega_{21}^1\omega^1] + 3\beta[\omega^2\omega_2^1],\\
D\omega_{21}^1 &= [\omega_2^2\omega_{21}^1] + [\omega_2^1\omega_{12}^2] + 3\alpha[\omega_1^2\omega_2^1] + 2\beta[\omega_{21}^1\omega^1] +\\
&\quad + \alpha[\omega_{12}^2\omega^1] + 2\beta_1[\omega^1\omega_2^1] + \alpha_2[\omega^1\omega_1^2] + 2\gamma[\omega^2\omega^1],\\
D\omega_{12}^2 &= [\omega_1^1\omega_{12}^2] + [\omega_1^2\omega_{21}^1] + 3\beta[\omega_2^1\omega_1^2] + 2\alpha[\omega_{12}^2\omega^2] +\\
&\quad + \beta[\omega_{21}^1\omega^2] + 2\alpha_2[\omega^2\omega_1^2] + \beta_1[\omega^2\omega_2^1] + 2\varepsilon[\omega^1\omega^2],
\end{aligned} \tag{A}$$

where along a fixed integral curve of equation (1) (respectively (2)) the equalities

$$\omega^1 = \omega_2^1 = 0, \tag{3}$$

respectively,

$$\omega^2 = \omega_1^2 = 0, \tag{4}$$

will be identically satisfied; and when a point on S is fixed—the equalities

$$\omega^1 = \omega^2 = 0. \tag{5}$$

As a consequence of (A), the systems (3), (4), and (5) are completely integrable. The representation of the group G mentioned above will be a representation in the space of first integrals of the completely integrable system

$$\omega^1 = \omega^2 = \omega_2^1 = \omega_1^2 = 0. \tag{6}$$

From the differential relations following from (A), it follows that α and β are relative invariants which vanish only simultaneously; moreover, when $\alpha = \beta = 0$, $\alpha_2 = \beta_1 = 0$.

Let $\alpha = \beta = 0$. In this case the following facts hold:

1) On the plane S a certain projective connection arises.

2) Equations (1) and (2) take the form:

$$y'' = A(y')^3 + B(y')^2 + Cy' + D, \tag{7}$$

$$x'' = -D(x')^3 - C(x')^2 - Bx' - A, \tag{8}$$

where A, B, C, D are arbitrary functions of x and y. It is easy to see that (7) and (8) are different forms of writing one and the same equation, and, consequently, their integral curves coincide.

3) If $\alpha = \beta = \gamma = \varepsilon = 0$, then a projective geometry arises on the plane S; the family of integral curves is equivalent to a rectilinear net—the set of lines of the projective plane. In the general case, when $\alpha \neq 0,\ \beta \neq 0$, introducing invariant dependences between the secondary parameters, one can express the forms $\omega_1^1, \omega_2^2, \omega_{21}^1, \omega_{12}^2$ through the principal forms $\omega^1, \omega^2, \omega_2^1, \omega_1^2$, which in this case will satisfy the following structural equations:

$$\begin{aligned}
D\omega^1 &= [\omega^2\omega_2^1] + 2[\omega_1^2\omega^1] + \beta_4[\omega_2^1\omega^1],\\
D\omega^2 &= [\omega^1\omega_1^2] + 2[\omega_2^1\omega^2] + \alpha_3[\omega_1^2\omega^2],\\
D\omega_2^1 &= (\alpha_3 - 2)[\omega_2^1\omega_1^2] + \alpha_2[\omega_2^1\omega^2] + \tfrac{1}{3}(\beta_1 + \beta_{24})[\omega_2^1\omega^1]\\
&\quad + \tfrac{2}{3}(\alpha_2 - \alpha_{13})[\omega^1\omega_1^2] + \tfrac{2}{3}(2\alpha_{12} - \beta_{22})[\omega^2\omega^1],\\
D\omega_1^2 &= (\beta_4 - 2)[\omega_1^2\omega_2^1] + \beta_1[\omega_1^2\omega^1] + \tfrac{1}{3}(\alpha_2 + \alpha_{13})[\omega_1^2\omega^2]\\
&\quad + \tfrac{2}{3}(\beta_1 + \beta_{24})[\omega^2\omega_2^1] + \tfrac{2}{3}(2\beta_{21} - \alpha_{11})[\omega^1\omega^2].
\end{aligned} \tag{B}$$

Let us note that all coefficients in (B) are absolute invariants.

We write the principal forms $\omega^1, \omega^2, \omega_2^1, \omega_1^2$ in the form

$$\begin{aligned}
\omega^1 &= \xi(dy - y'\,dx), \quad \omega_2^1 = \rho\{dy - y'\,dx + \lambda[dy' - f(x,y,y')\,dx]\},\\
\omega^2 &= \eta(dx - x'\,dy), \quad \omega_1^2 = \sigma\{dx - x'\,dy + \mu[dx' - \varphi(x,y,x')\,dy]\},
\end{aligned} \tag{9}$$

where $\xi, \eta, \rho, \sigma, \lambda, \mu$ are functions of the principal parameters x, y, x', y'.

Externally differentiating (9) and comparing the results with (B), we obtain:

$$\xi = -\tfrac{1}{3}\left(f_{y'y'} + 4kx'f_{y'} + 6k^2(x')^2 f + 6k^2 y'\varphi + 2k\varphi_{x'}\right),$$

$$\eta = -\tfrac{1}{3}\left(\varphi_{x'x'} + 4ky'\varphi_{x'} + 6k^2(y')^2\varphi + 6k^2 x' f + 2kf_{y'}\right),$$

$$\alpha_3 = \frac{\xi}{\eta}\left(\frac{\eta_{x'}}{\eta^k} - y'\right), \qquad \beta_4 = \frac{\eta}{\xi}\left(\frac{\xi_{y'}}{\xi^k} - x'\right) \qquad \left(k = \frac{1}{1 - x'y'}\right),$$

$$\beta_1 = -\frac{2\eta_1}{\eta} - \frac{k}{\xi}\left[k\left(fx'^2 + y'\varphi - \frac{x'\eta\xi_2}{k\xi}\right) - \frac{1}{\xi}\left(\xi_x x' + \xi_y + 2k\eta\varphi + \xi_{y'} fx' - \frac{\xi_{y'}\xi_2\eta}{k^2\xi}\right)\right],$$

$$\alpha_2 = -\frac{2\xi_2}{\xi} - \frac{k}{\eta}\left[k\left(\varphi y'^2 + x'f - \frac{y'\xi\eta_1}{k\eta}\right) - \frac{1}{\eta}\left(\eta_x + \eta_y y' + \eta_{x'}\varphi y' + 2k\xi f - \frac{\eta_{x'}\eta_1\xi}{k^2\eta}\right)\right],$$

$$^2/_3\left(\alpha_{13} + \alpha_2\right) = -\frac{\xi_{2x'}}{k\eta} + \frac{\xi_2}{k\eta\xi}\left(2k\eta + k\xi y' - \frac{\xi\eta_{x'}}{\eta}\right),$$

$$^2/_3\left(\beta_{24} + \beta_1\right) = -\frac{\eta_{1y'}}{k\xi} + \frac{\eta_1}{k\eta\xi}\left(2k\xi + k\eta x' - \frac{\eta\xi_{y'}}{\xi}\right), \tag{10}$$

$$^1/_3\left(2\alpha_{12} - \beta_{22}\right) = \frac{k}{2\eta\xi}\left(\xi_{2x} + \xi_{2y}y' + \xi_{2x'}\varphi y' + \xi_{2y'}f - \frac{\xi_{2x'}\eta_1\xi}{k^2\eta}\right) - \frac{\xi_2^2}{\xi^2} -$$

$$-\frac{k\xi_2}{2\eta\xi}\left[k\left(\varphi y'^2 + fx' - \frac{y'\xi\eta_1}{k\eta}\right) - \frac{1}{\eta}\left(\eta_x + \eta_y y' + \eta_{x'}\varphi y' + 2k\xi f - \frac{\eta_{x'}\eta_1\xi}{k^2\eta}\right)\right],$$

$$^1/_3\left(2\beta_{21} - \alpha_{11}\right) = \frac{k}{2\eta\xi}\left(\eta_{1x}x' + \eta_{1y} + \eta_{1y'}fx' - \frac{\eta_{1y'}\xi_2\eta}{k^2\xi}\right) - \frac{\eta_1^2}{\eta^2}$$

$$-\frac{k\eta_1}{2\xi\eta}\left[k\left(fx'^2 + \varphi y' - \frac{x'\eta\xi_2}{k\xi}\right) - \frac{1}{\xi}\left(\xi_x x' + \xi_y + 2k\eta\varphi + \xi_{y'}fx' - \frac{\xi_{y'}\xi_2\eta}{k^2\xi}\right)\right],$$

where, for brevity, it is denoted:

$$\eta_1 = \frac{k}{3\xi}\left[2\left(\xi_x + \xi_y y' + 2k\eta\varphi y' + \xi_{y'}f\right) - \left(\eta_x x' + \eta_y + \eta_{x'}\varphi + 2k\xi f x'\right)\right],$$

$$\xi_2 = \frac{k}{3\eta}\left[2\left(\eta_x x' + \eta_y + \eta_{x'}\varphi + 2k\xi f x'\right) - \left(\xi_x + \xi_y y' + 2k\eta\varphi y' + \xi_{y'}f\right)\right]. \tag{11}$$

Formulas (10), (11) will be the solution of the problem in general form, since they make it possible, for the given functions f and φ, to find directly by computation the invariants of the structural equations (B).

In conclusion, let us consider one of the simplest special cases, which we single out as follows: let us require that the greatest possible number of invariants vanish, and that the remaining ones become constants, with symmetry necessarily preserved. Such a system of invariants turns out to be possible, and the corresponding structural formulas, after simplification, take the form:

$$D\omega^1 = [\omega^2\omega_2^1] + 2[\omega_1^2\omega^1] + [\omega_2^1\omega^1],$$

$$D\omega^2 = [\omega^1\omega_1^2] + 2[\omega_2^1\omega^2] + [\omega_1^2\omega^2], \tag{C}$$

$$D\omega_1^2 = [\omega_2^1\omega^2] + 3[\omega_1^2\omega^1] + [\omega_1^2\omega^2] + 2[\omega^2\omega_1^2] + 2[\omega^2\omega^1],$$

$$D\omega_2^1 = [\omega_1^2\omega^1] + 3[\omega_2^1\omega^2] + [\omega_2^1\omega^1] + 2[\omega^1\omega_1^2] + 2[\omega^1\omega^2].$$

It is easily verified that (C) are the structural equations of a four-parameter Lie group, isomorphic and similar to a subgroup of the affine group of the plane:

$$\tilde{u} = \frac{c_1}{2}\left[(1 - c_1 - c_2)u + (1 + c_1 - c_2)v + c_3\right],$$

$$\tilde{v} = \frac{c_1}{2}\left[(1 + c_1 + c_2)u + (1 - c_1 + c_2)v + c_4\right].$$

Equations (1) and (2) will take the following form in these coordinates:

$$u'' = \frac{3}{4}(1 + u')^3, \tag{12}$$

$$v'' = \frac{3}{4}(1 + v')^3. \tag{13}$$

On the right-hand sides of (12) and (13) there stand polynomials of the third degree in the first derivatives. Since this form of a differential equation of the second order is invariant under arbitrary analytic transformations of the plane, it may be asserted that, in the case under consideration, the right-hand sides of (1) and (2) are polynomials of the third degree in the first derivatives.

A deeper investigation makes it possible to establish that the correspondence between the plane S and the plane with an affine subgroup can be realized in infinitely many ways. We give an example.

It follows from (C) that the parameters may be chosen so that

$$\omega^1 + \omega^2 - \omega_1^2 - \omega_2^1 = -d\ln t,$$

$$p(\omega^1 + \omega^2) + q(\omega_1^2 + \omega_2^1) + r(2\omega^2 + \omega_1^2) = t\,dw,$$

$$\omega^2 = \left(-\frac{q}{r}t - \frac{w}{r}t^2\right) du + t^2\,dv, \qquad \omega^1 + \omega^2 = t\,du,$$

where p, q, r are arbitrary constants, $r \neq 0$. The plane u, v is transformed affinely, since integration of the corresponding equations gives

$$\tilde{u} = c_1 u + c_2, \qquad \tilde{v} = \frac{c_1 c_3}{r} u + c_1^2 v + c_4.$$

Equations (1) and (2), for this choice of the parameters, take the form:

$$v'' = \frac{p+q}{r}, \qquad v'' = \frac{p+q}{r} + 3.$$

For $p + q = -3r$ or $p = -q$, the corresponding family of integral curves will be mapped onto a set of straight lines in the plane u, v. Consequently, in the case under consideration, each of the families of integral curves separately is equivalent to a rectilinear net. In all other cases the families of integral curves are mapped into families of parabolas. It is interesting to note that the planes u, v corresponding to different p, q, r turn out to be birationally equivalent, since they pass into one another under transformations of the form

$$\bar{u} = u, \qquad \bar{v} = v + Lu^2,$$

where L is a constant.

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CITED LITERATURE

[1] E. Cartan, Bull. Soc. Math., **52**, 205 (1924).
[2] J. Dubourdieu, *Questions topologiques de géométrie différentielle*, Paris, 1936.
[3] A. M. Vasil' ev, DAN, **79**, No. 1 (1951).
[4] G. F. Laptev, Tr. Mosk. matem. obshch., **2**, 275 (1953).
[5] S. P. Finikov, *The Method of Cartan' s Exterior Forms*, Moscow-Leningrad, 1948.
[6] M. V. Vasil' eva, Matem. sborn., **36**, issue 1, 57 (1955).