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V. A. Gaukhman

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Abstract

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V. A. Gaukhman

On Geodesic Surfaces in Manifolds of Affine Connection with a Complex Structure

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In the present work the investigation is local in character. Let $\mathfrak{M}_{2n}$ be a $2n$-dimensional manifold with a complex analytic structure, on which an almost complex affine connection without torsion is given (^{1}).

For frames adapted to the complex structure, the equations of infinitesimal displacements have the form:

$$\begin{aligned} dM &= \omega^i e_i + \tilde{\omega}^i e_{n+i}, \\ de_i &= \omega_i^j e_j + \tilde{\omega}_i^j e_{n+j}, \qquad (i, j, k = 1, \dots, n). \\ de_{n+i} &= -\tilde{\omega}_i^j e_j + \omega_i^j e_{n+j}. \end{aligned}$$

The structure equations of an almost complex connection are written in the form:

$$\begin{aligned} D\omega^i &= [\omega^j \omega_j^i] - [\tilde{\omega}^j \tilde{\omega}_j^i], \\ D\tilde{\omega}^i &= [\tilde{\omega}^j \omega_j^i] + [\omega^j \tilde{\omega}_j^i], \\ D\omega_i^j &= [\omega_i^k \omega_k^j] - [\tilde{\omega}_i^k \tilde{\omega}_k^j] + [\omega^k, R_{jkl}^i \omega^l + \widetilde{R}_{jkl}^i \tilde{\omega}^l] - [\tilde{\omega}^k, S_{jkl}^i \omega^l + \widetilde{S}_{jkl}^i \tilde{\omega}^l], \\ D\tilde{\omega}_i^j &= [\omega_i^k \tilde{\omega}_k^j] + [\tilde{\omega}_i^k \omega_k^j] + [\tilde{\omega}^k, R_{jkl}^i \omega^l + \widetilde{R}_{jkl}^i \tilde{\omega}^l] + [\omega^k, S_{jkl}^i \omega^l + \widetilde{S}_{jkl}^i \tilde{\omega}^l]. \end{aligned}$$

Denoting $\Omega^i = \omega^i + \sqrt{-1}\,\tilde{\omega}^i$; $\Omega_i^j = \omega_i^j + \sqrt{-1}\,\tilde{\omega}_i^j$, we obtain:

$$D\Omega^i = [\Omega^k \Omega_k^i], \tag{1}$$

$$D\Omega_i^j = [\Omega_i^k \Omega_k^j] + K_{jkl}^i [\Omega^k \Omega^l] + L_{jkl}^i [\Omega^k \overline{\Omega}^l], \tag{2}$$

where

$$K_{jkl}^i = \frac{1}{2}\left[(R_{j[kl]}^i + \widetilde{S}_{j[kl]}^i) + \sqrt{-1}\,(S_{j[kl]}^i - \widetilde{R}_{j[kl]}^i)\right],$$

$$L^i_{jkl} = \frac{1}{2}[(R^i_{jkl} - \widetilde{S}^i_{jkl}) + \sqrt{-1}\,(S^i_{jkl} + \widetilde{R}^i_{jkl})].$$

Exterior differentiation of equations (1) gives:

$$K^i_{(jkl)} = 0; \qquad L^i_{[jk]l} = 0. \tag{3}$$

Let C_n be a complex analytic manifold of complex dimension n (see (^{2})), whose equations of infinitesimal displacements have the form:

$$dM = \Omega^i E_i,$$

$$dE_i = \Omega^j_i E_j.$$

Then (1), (2) are the structure equations of this complex manifold.

Every $2r$-dimensional analytic surface* in $\mathfrak{M}_{2n}$ is given by equations of the form:

$$\omega^\lambda = a^\lambda_\tau \omega^\tau - \tilde{a}^\lambda_\tau \tilde{\omega}^\tau, \qquad \tilde{\omega}^\lambda = \tilde{a}^\lambda_\tau \omega^\tau + a^\lambda_\tau \tilde{\omega}^\tau, \tag{4}$$

$$(\tau = 1, \dots, r;\ \lambda = r+1, \dots, n),$$

or, what is the same, by equations of the form

$$\Omega^\lambda = A^\lambda_\tau \Omega^\tau,$$

where

$$A^\lambda_\tau = a^\lambda_\tau + \sqrt{-1}\,\tilde{a}^\lambda_\tau.$$

It follows from this that a $2r$-dimensional analytic surface in $\mathfrak{M}_{2n}$ is the image of an arbitrary complex analytic surface of complex dimension r in C_n.

Let $\mathfrak{N}_2 \subset \mathfrak{M}_{2n}$ be a two-dimensional analytic and totally geodesic (see (3)) surface. $\mathfrak{N}_2$ can be given by equations (4), in which $r = 1$. The condition that $\mathfrak{N}_2$ be totally geodesic has the form:

$$da^\lambda_1 + a^\mu_1 \omega^\lambda_\mu - \tilde{a}^\mu_1 \tilde{\omega}^\lambda_\mu - a^\lambda_1 \omega^1_1 + \tilde{a}^\lambda_1 \tilde{\omega}^1_1 + \omega^\lambda_1 - a^\lambda_1 (a^\mu_1 \omega^1_\mu - \tilde{a}^\mu_1 \tilde{\omega}^1_\mu) +$$

$$+ \tilde{a}^\lambda_1 (\tilde{a}^\mu_1 \omega^1_\mu + a^\mu_1 \tilde{\omega}^1_\mu) = 0,$$

$$d\tilde{a}^\lambda_1 + \tilde{a}^\mu_1 \omega^\lambda_\mu + a^\mu_1 \tilde{\omega}^\lambda_\mu - \tilde{a}^\lambda_1 \omega^1_1 - a^\lambda_1 \tilde{\omega}^1_1 - \tilde{\omega}^\lambda_1 - \tilde{a}^\lambda_1 (a^\mu_1 \omega^1_\mu - \tilde{a}^\mu_1 \tilde{\omega}^1_\mu) -$$

$$- a^\lambda_1 (\tilde{a}^\mu_1 \omega^1_\mu + a^\mu_1 \tilde{\omega}^1_\mu) = 0. \tag{5}$$

Equations (5) can be written more compactly in the form

$$dA^\lambda_1 + A^\mu_1 \Omega^\lambda_\mu - A^\lambda_1 \Omega^1_1 + \Omega^\lambda_1 - A^\lambda_1 A^\mu_1 \Omega^1_\mu = 0. \tag{6}$$

But (6) means that the line $\Omega^\lambda = A_1^\lambda \Omega'$ is geodesic in C_n, i.e. that along this line $d^2M = \Lambda \cdot dM$. Consequently, a two-dimensional analytic totally geodesic surface in $\mathfrak{M}_{2n}$ is the image of a geodesic line in C_n.

Lemma. *In the space C_n, through every point and in every complex direction there passes, moreover uniquely, a geodesic line.*

Proof. We exteriorly differentiate the system (6). In view of (2), (3), (6), we obtain an identity. Hence the system (6) is completely integrable, which proves the lemma.

It follows immediately from the lemma that in $\mathfrak{M}_{2n}$, through any point and in any two-dimensional analytic direction, there passes a unique analytic totally geodesic surface.

Now let x_0 be a point of the manifold $\mathfrak{M}_{2n}$ and let π_{2r} be some analytic $2r$-dimensional direction at x_0. In C_n this direction corresponds to some complex direction Π_r of complex dimension r, issuing from a certain point $X_0 \in C_n$. Consider in C_n the surface formed by all complex geodesic lines passing through the point X_0 and tangent at this point to the plane Π_r. In $\mathfrak{M}_{2n}$ there corresponds to it an analytic surface. Obviously, this surface is formed by geodesic lines tangent at the point x_0 to the plane π_{2r}, i.e. this surface is geodesic at the point x_0 (see (3)). We have proved:

Theorem. *Let $\mathfrak{M}_{2n}$ be a $2n$-dimensional manifold with a complex structure and with an almost complex torsion-free affine connection. Let*

x_0 is an arbitrary point of $\mathfrak{M}_{2n}$; π_{2r} is an arbitrary $2r$-dimensional analytic subspace of the tangent space at the point x_0. Then the $2r$-dimensional surface formed by all geodesic lines passing through the point x_0 and tangent at this point to the plane π_{2r} is analytic. If $r = 1$, then the corresponding two-dimensional surface is also totally geodesic.

Corollary. If an analytic two-dimensional surface $\mathfrak{M}_2 \subset \mathfrak{M}_{2n}$ contains one geodesic line, then it is totally geodesic.

The corollary becomes obvious if one observes that through every line in $\mathfrak{M}_{2n}$ there passes a unique two-dimensional analytic surface.

In conclusion we note that the following assertion can be proved: if, on a manifold $\mathfrak{M}_{2n}$ with complex structure, a torsion-free affine connection is given such that all the surfaces discussed in the theorem are analytic, then this connection is projectively equivalent to an almost complex torsion-free connection, i.e., on

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[*] A subspace of the tangent space to $\mathfrak{M}_{2n}$ at the point x is called analytic if it is invariant under the automorphism $I : e_i \to e_{n+i},\ e_{n+i} \to -e_i$ (see (2)). A surface is called analytic at a point $x \in \mathfrak{M}_{2n}$ if its tangent plane at this point is analytic. A surface is called analytic if all its tangent planes are analytic.

$\mathfrak{M}_{2n}$ there exists an almost complex torsion-free connection having the same geodesic lines as the original connection.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.