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Abstract

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AN INTERESTING CLASS OF QUASILINEAR SYSTEMS

(Presented by Academician I. G. Petrovskii, 17 III 1961)

In this paper I wish to draw attention to one class of differential equations which encompasses a number of important equations of mathematical physics and is convenient for the construction of a mathematical theory.

The equations for reversible processes belonging to the class described have the form

$$\frac{\partial L_{q_i}}{\partial t} + \sum_j \frac{\partial L^j_{q_i}}{\partial x_j} = 0. \qquad (1)$$

Here $L = L(q_1, q_2, \dots, q_n)$, $L^j = L^j(q_1, q_2, \dots, q_n)$.

The following can be reduced to this class:

1. The variational Lagrange equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t^{(k)}}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial \mathcal{L}}{\partial u_{x_1}^{(k)}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \mathcal{L}}{\partial u_{x_2}^{(k)}}\right) = 0,$$

$$\mathcal{L} = \mathcal{L}(u_t^{(1)}, u_{x_1}^{(1)}, u_{x_2}^{(1)}, u_t^{(2)}, \dots, u_{x_2}^{(n)}).$$

To perform the reduction one must set

$$q_{3k} = u_t^{(k)}, \qquad q_{3k-1} = -\frac{\partial \mathcal{L}}{\partial u_{x_1}^{(k)}}, \qquad q_{3k-2} = -\frac{\partial \mathcal{L}}{\partial u_{x_2}^{(k)}},$$

$$L = \mathcal{L} - \sum_k \left[u_{x_1}^{(k)} \frac{\partial \mathcal{L}}{\partial u_{x_1}^{(k)}} + u_{x_2}^{(k)} \frac{\partial \mathcal{L}}{\partial u_{x_2}^{(k)}}\right], \qquad L^1 = \sum_k u_t^{(k)} \frac{\partial \mathcal{L}}{\partial u_{x_1}^{(k)}}, \qquad L^2 = \sum_k u_t^{(k)} \frac{\partial \mathcal{L}}{\partial u_{x_2}^{(k)}}.$$

2. The differential equations of crystal optics.
3. The equations of gas dynamics

$$\frac{\partial \rho u_1}{\partial t}+\frac{\partial(\rho u_1^2+p)}{\partial x_1}+\frac{\partial \rho u_1 u_2}{\partial x_2}+\frac{\partial \rho u_1 u_3}{\partial x_3}=0,$$

$$\frac{\partial \rho u_2}{\partial t}+\frac{\partial \rho u_2 u_1}{\partial x_1}+\frac{\partial(\rho u_2^2+p)}{\partial x_2}+\frac{\partial \rho u_2 u_3}{\partial x_3}=0,$$

$$\frac{\partial \rho u_3}{\partial t}+\frac{\partial \rho u_3 u_1}{\partial x_1}+\frac{\partial \rho u_3 u_2}{\partial x_2}+\frac{\partial(\rho u_3^2+p)}{\partial x_3}=0,$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_1}{\partial x_1}+\frac{\partial \rho u_2}{\partial x_2}+\frac{\partial \rho u_3}{\partial x_3}=0,$$

$$\frac{\partial \rho\left(E+\frac{u_1^2+u_2^2+u_3^2}{2}\right)}{\partial t}+\frac{\partial \rho u_1\left(E+\frac{p}{\rho}+\frac{u_1^2+u_2^2+u_3^2}{2}\right)}{\partial x_1}+$$

$$+\frac{\partial \rho u_2\left(E+\frac{p}{\rho}+\frac{u_1^2+u_2^2+u_3^2}{2}\right)}{\partial x_2}+\frac{\partial \rho u_3\left(E+\frac{p}{\rho}+\frac{u_1^2+u_2^2+u_3^2}{2}\right)}{\partial x_3}=0.$$

For the reduction it is necessary to set

$$q_1=-\frac{u_1}{T},\qquad q_2=-\frac{u_2}{T},\qquad q_3=-\frac{u_3}{T},$$

$$q_4=S-\frac{E+\frac{p}{\rho}-\frac{u_1^2+u_2^2+u_3^2}{2}}{T},\qquad q_5=\frac{1}{T},$$

$$L=-\frac{p}{T},\qquad L^1=-\frac{u_1 p}{T},\qquad L^2=-\frac{u_2 p}{T},\qquad L^3=-\frac{u_3 p}{T}.$$

The possibility of such a reduction follows from the thermodynamic identity $dE+p\,d\frac{1}{\rho}=T\,dS$.

It is easy to verify that on smooth solutions of system (2) the conservation law

$$\frac{\partial\left(\sum_i q_i L_{q_i}-L\right)}{\partial t}+\sum_j \frac{\partial\left(\sum_i q_i L^j_{q_i}-L^j\right)}{\partial x_j}=0$$

is satisfied.

In most concrete examples it represents a conservation law of energy or entropy.

Systems of type (1) can be rewritten in the form

$$\sum_k L_{q_{i_q}k} \frac{\partial q_k}{\partial t} + \sum_{j,k} L^j_{q_{i_q}k} \frac{\partial q_k}{\partial x_j} = 0, \tag{2}$$

from which it follows that, for convex $L(q_1, q_2, \dots, q_n)$, they are a natural nonlinear generalization of Friedrichs symmetric systems ([1]). Differentiating system (2) several times with respect to x_i and t, we obtain equations satisfied by certain derivatives of q_i. The symmetry of the coefficient matrices of these equations makes it possible without difficulty to write energy integrals for the derivatives. These integrals make it possible to prove the correctness of system (1), for example, by the same method as was done in the work of I. G. Petrovsky ([2]).

Equations for irreversible processes are obtained from equations (1) by adding dissipative terms:

$$\frac{\partial L_{q_i}}{\partial t} + \sum_j \frac{\partial L^j_{q_i}}{\partial x_j} = \sum_{j,k,s} \frac{\partial}{\partial x_j} b^{js}_{ik} \frac{\partial q_k}{\partial x_s}. \tag{3}$$

The matrix $\|b^{js}_{ik}\|$ must be symmetric in j, s and in i, k and positive definite. The symmetry condition is a consequence of Onsager' s conditions for irreversible processes ([3]).

It turns out that the form (3) of the equations for dissipative processes is very convenient in investigating the structure of shock waves. This was shown by me in work ([4]), where I reduced the study of the structure to ordinary equations of the form

$$\Lambda q_i = D_{\sigma_i}, \qquad dq_i = \sigma_i \, d\tau, \tag{4}$$

$$D = \sum b_{ik} \sigma_i \sigma_k, \qquad \Lambda = \Lambda(q_1, q_2, \dots, q_n)$$

and gave a geometric interpretation for these equations. It was then used in works ([5–7]), which made it possible to reveal a number of unexpected facts about the behavior of solutions of quasilinear systems.

During the discussion of this work at L. I. Sedov' s seminar at Moscow State University, my attention was drawn to the fact that the form (4) for equations describing the structure of waves in magnetic hydrodynamics had been noted by Germain (8). However, the geometric interpretation of equations (4) remained unnoticed by Germain.

The singling out of the classes of equations described here arose for me as a consequence of studying the connection between thermodynamics and the correctness of the equations of mathematical physics, begun in the work (9).

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Note: Figure translations are in progress. See original paper for figures.