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Abstract

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A. M. MOLCHANOV

SEPARATION OF MOTIONS AND ASYMPTOTIC METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS

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At the present time the most effective device in the asymptotic theory of nonlinear oscillations is the averaging method ([1]). In the present note the possibility is discussed of another point of view on asymptotic methods–the point of view of separating motions into fast and slow (evolutionary) ones. As will be shown, the averaging method, as well as the idea of studying systems with rapidly rotating phases ([1,2]), which is closely related to it, are special cases of the method of separation of motions.

Passing to the formulation of the idea of separation of motions, let us consider in parallel the systems of equations of unperturbed motion

$$du/dt = U(u) \tag{1}$$

and of perturbed motion

$$dv/dt = U(v) + V(v), \tag{2}$$

where the perturbing function V admits an asymptotic expansion in powers of a small parameter $V(\varepsilon, v) = \varepsilon V_1(v) + \varepsilon^2 V_2(v) + \ldots$. Denote by

$$u = u(w, t) \tag{3}$$

the solution of system (1) satisfying the initial data $u|_{t=0} = u(w, 0) = w$.

It is not difficult to show that the solution of equation (2) can also be written in the form (3), but then one must assume that w is not constant, but depends on t–a device analogous to the method of variation of arbitrary constants in linear equations. The resulting equation for w, generally speaking, explicitly contains time. In the linear case, in particular, secular terms may appear. It is clear that the investigation is substantially simplified if for w one obtains an autonomous equation, i.e., one not explicitly containing time. It is natural to call this equation evolutionary, since it describes the slow changes of the system that do not depend on its principal, fast motion. In this case we shall say of system (2) that it admits a separation of motions.

Systems admitting a separation of motions play, among general systems of the form (2), a role analogous to that of diagonal systems among linear ones. Just as systems of linear equations can be brought to diagonal form by a linear change of variables, the nonlinear system (2) can be brought by a nonlinear change of variables to a form admitting separation of motions. The exceptional case is that of degeneracy, corresponding to the case of Jordan form in linear equations.

Our immediate goal is to find the conditions under which the system admits separation of motions. Substituting (3) into (2), we have

$$\frac{\partial u}{\partial t} + \frac{\delta u}{\delta w}\frac{dw}{dt} = U(u) + V(u).$$

Here and below, the symbol $\delta u/\delta w$ and analogous ones denote the matrix whose elements are the partial derivatives of the components of the vector u with respect to the components of the vector w. By the definition of the function $u(w,t)$, its partial derivative with respect to t is equal to $U(u)$. Therefore the first terms on both sides of the equality cancel one another, and, after multiplying on the left by $(\delta u/\delta w)^{-1}$, we obtain:

$$dw/dt = W(w,t), \tag{4}$$

where the notation

$$W = (\delta u/\delta w)^{-1} V \tag{5}$$

has been introduced.

By the definition of systems admitting separation of motions, the right-hand side in (4) must not contain the time explicitly. Simple calculations show that

$$\frac{\partial W}{\partial t} = \left(\frac{\delta u}{\delta w}\right)^{-1} \left[\frac{\delta V}{\delta u}U - \frac{\delta U}{\delta u}V\right]. \tag{6}$$

Therefore a necessary and sufficient condition for separation of motions is the equality

$$\frac{\delta V}{\delta u}U - \frac{\delta U}{\delta u}V = 0. \tag{7}$$

If it is satisfied, then the solution of equation (2) is given by the formula

$$v = u(w(t), t), \tag{8}$$

where $w(t)$ is a solution of equation (4), which in this case takes the form

$$dw/dt = V(w). \tag{9}$$

This follows at once from equality (5) for $t = 0$, since then $\delta u/\delta w = E$. But since W does not depend on t, the equality $W = V(W)$ holds identically.

Thus, when condition (7) is fulfilled, in order to solve equation (2) it is sufficient to solve independently equations (1) and (9), and then, in the solution of one of these equations (either one, in view of the complete symmetry of condition (7)), substitute, instead of the initial data (see (8)), the solution of the other equation. This circumstance seems to be sufficient justification for introducing the term "separation of motions." We note that if U and V are linear functions of their argument, then condition (7) is simply the condition that the matrices U and V commute. Therefore the left-hand side of condition (7) is a natural generalization of the concept of a commutator to the case of nonlinear operators.

Let us turn to the proof of the fact that, by a change of variables, system (2) can be brought to a form admitting separation of motions. If up to now the smallness of the parameter ε has played no role, it now becomes decisive. Namely, the existence will be proved of an asymptotic series $y = v + \varepsilon Q_1(v) + \varepsilon^2 Q_2(v) + \cdots$ such that the equation for y already admits separation of motions. Straightforward but cumbersome calculations show that for y one obtains an equation analogous to the equation for v, with the same leading term, as expected:

$$dy/dt = U(y) + \varepsilon Y_1(y) + \varepsilon^2 Y_2(y) + \cdots. \tag{10}$$

It can be verified that the coefficient Q_n to be determined enters into Y_n in the following way:

$$Y_n = \widetilde{Y}_n + \frac{\delta Q_n}{\delta y} U - \frac{\delta U}{\delta y} Q_n, \tag{11}$$

where $\widetilde{Y}_n$, apart from the prescribed functions $V_1, \dots, V_n$, depends only on the preceding $Q_1, \dots, Q_{n-1}$. $\widetilde{Y}_1$, in particular, is simply $V_1(y)$. Therefore, in determining the next Q_n, $\widetilde{Y}_n$ may be regarded as a known function of y. We must choose the coefficients Q_n so that equation (10) admits a separation of motions. As we saw above, this is equivalent to the requirement that the functions Y_n commute with U in the sense of condition (7). Introducing the notation

$$\mathcal{L}_U(Y) = \frac{\delta Y}{\delta u} U - \frac{\delta U}{\delta u} Y, \tag{12}$$

we see that the problem of determining Q_n reduces to the problem of decomposing the known function $\widetilde{Y}_n$ into the sum of two terms, one of which belongs to the range of the linear operator $\mathcal{L}_U$, while the other is annihilated by this operator:

$$\widetilde{Y}_n = -\mathcal{L}_U(Q_n) + Y_n, \qquad \mathcal{L}_U(Y_n) = 0. \tag{13}$$

Strictly speaking, expression (12) is not yet an operator, since in order to specify an operator completely it is necessary to indicate its domain of definition. The choice of the domain of definition is dictated by the set of functions $\widetilde{Y}_n$ that must be decomposed into the sum (13). If this choice has somehow been made, the question of the possibility of the decomposition (13) reduces to the question of the absence, for the operator $\mathcal{L}_U$, of a Jordan cell corresponding to the zero eigenvalue. We shall not now discuss the possibility of such a representation, but shall turn to the consideration of one practically important case in which it is possible not only to prove the existence of the decomposition, but actually to construct it.

This construction is based on another interpretation of the decomposition (13) —an interpretation obtained as follows. Formula (5), with each function $V(u)$, uniquely associates a function of w and t, which is the result of parallel transport of $V(u)$ along the trajectories of the unperturbed motion $u(w,t)$. The decomposition (13) gives, among such functions, a decomposition admitting a very simple interpretation. It is the decomposition of any function of the form (5) into the sum of two: a function integrable along trajectories (i.e. representable as a partial derivative with respect to t of a function of the same form), and a function that does not change under a shift along trajectories. Such an interpretation follows directly from formulas (5), (6), and (13). Let us note that, by virtue of the same formulas, and also of the equality $\delta u/\delta w = E$ at $t = 0$, which follows from (3), the decomposition of functions of the form (5), upon substituting $t = 0$, gives the decomposition (13).

The interpretation just analyzed is general and is always suitable, while in one special case important for applications it directly leads to an effective solution of the problem. This is the case when the shift of $\widetilde{Y}_n$ along trajectories gives rise to an almost periodic function of t whose frequencies do not accumulate at zero. (In what follows, precisely such functions are meant.) Then the problem reduces simply to extracting from the function its mean value, since it is not difficult to verify that equality of the mean value to zero is a necessary and sufficient condition for the integrability (in the sense of remaining within the class) of such functions. Therefore the decomposition of a function into a constant part and a function with zero mean value coincides with the decomposition we need. Without going into the details of the exposition, let us give the final result (in the formulas below $u = u(w,t)$, and integration with respect to t is performed along a trajectory, i.e. with fixed w):

$$Y_n(w) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{\delta u}{\delta w}\right)^{-1} \widetilde{Y}_n(u)\, dt, \tag{14}$$

$$Q_n(w) = \lim_{T\to\infty} \frac{1}{T} \int_0^T (T-t) \left(\frac{\partial u}{\partial w}\right)^{-1} [\widetilde{Y}_n(u) - Y_n(u)]\, dt. \tag{15}$$

The formulas obtained are somewhat simplified if $\widetilde{Y}_n$ turns out to be a periodic function of t. In this case it is obviously sufficient to take the average over the period T, even if this period depends on w.

Let us note that the ambiguity in the choice of Q_n (since to Q_n one may add any term Q_n' for which $\mathcal{L}_U(Q_n') = 0$) is eliminated in formula (15) by the requirement that the mean of Q_n be equal to zero. This arbitrary choice, which is inessential for constructing the asymptotic theory, may be important in studying the convergence of asymptotic series. Such a method of constructing Q_n may prove not to be the most successful, although at first sight it is the one that most promotes the convergence of the series for y.

The second remark concerns the fact that the derivation of formulas (14) and (15) relied on the fact that the integrand is almost periodic in t. The factor $(\partial u/\partial w)^{-1}$ may fail to be almost periodic in the case of rapidly rotating phases. A certain modification of the derivation leads in this case to formulas analogous to (14) and (15).

Finally, the third remark concerns the possibility of generalizing the averaging formulas to the nonperiodic case by analytically continuing $u(w,t)$ to complex values of t, so that the integration is performed along some curve in the complex domain along which mean values have meaning. Thus, for example, if $U(u)$ is a linear operator with real eigenvalues, then formulas (14) and (15) give the desired expansion when integrating along the imaginary axis t. It would be interesting to find out whether such a generalization is illusory, or whether it substantially extends the class of problems (13) admitting solution by means of formulas of the type (14), (15). The example of the function $\operatorname{sh} t + \sin t$ is not very encouraging in this respect.

It is therefore interesting to try to find direct approaches to solving the expansion problem. We now turn to the discussion of one such possibility. The equations of the unperturbed motion look simplest if, as the new unknowns, one chooses a system of first integrals of equation (1). But since the number of first integrals is one less than the order of the system, one must adjoin to them one more independent function, which is naturally called the phase variable. Such variables can in any case be chosen in a neighborhood of any regular point of equation (1). It is not difficult to verify that in these variables the expansion problem leads to equations admitting integration in quadratures. The resulting solution contains, of course, arbitrary functions. Formally speaking, separation of motions takes place for any choice of these functions. However, the effectiveness of the asymptotic expansion depends essentially on the boundedness of the coefficients. The requirement that the coefficients be bounded makes it possible in some cases to eliminate the arbitrariness in the choice of the solution. Lack of space does not make it possible in this note to discuss in greater detail this interesting question, which is probably connected with questions of convergence of asymptotic expansions.

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[1] N. N. Bogolyubov, Yu. L. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1958. [2] V. M. Volosov, DAN, 133, No. 2 (1960).

Note: Figure translations are in progress. See original paper for figures.