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Abstract

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ON SOME THEOREMS OF PALEY-WIENER TYPE

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In the study of various questions connected with differential operators with constant coefficients, the usual method consists in applying the Fourier transform and considering the corresponding operators in the dual space. One of the spaces of this type most frequently encountered in applications is the space $\mathcal{H}$ of entire analytic functions $u(\xi) = u(\xi_1, \dots, \xi_n)$ of first order of growth of fixed type, quadratically summable in the real domain. If R_n is the real space of the variables $\xi = (\xi_1, \dots, \xi_n)$ and $\mathcal{M} \subset R_n$ is an arbitrary set of positive measure, then the expression

$$\|u\|_M = \left(\int_M |u(\xi)|^2 \, d\xi \right)^{1/2}$$

defines in the space $\mathcal{H}$ a norm weaker than the usual norm

$$\|u\| = \left(\int_{R_n} |u(\xi)|^2 \, d\xi \right)^{1/2}.$$

In the present note we establish a certain class of sets $\mathcal{M}$ that determine, in the above sense, a topology of the space $\mathcal{H}$ equivalent to the usual topology of a Hilbert space. The results obtained naturally generalize to a considerably broader class of dual spaces, but we shall not dwell here on these generalizations.

Basic notation. We shall denote by Ω a bounded domain in the space R^{n*} and by $H(\Omega)$ the totality of all square-summable functions $u(x)$ in Ω. The Fourier transforms of functions $u(x) \in H(\Omega)$ constitute the space $\mathcal{H}$, i.e. $\hat{u}(\xi) \in \mathcal{H}$ if and only if

$$\hat{u}(\xi) = \left(\frac{1}{2\pi} \right)^{n/2} \int_\Omega u(x) e^{i\langle x, \xi \rangle} \, dx, \qquad u(x) \in H(\Omega). \tag{1}$$

In the space $\mathcal{H}$ the topology is given in the natural way:

$$\|\hat{u}(\xi)\| = \left(\int_{R_n} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}. \tag{2}$$

By E we shall denote the unit sphere in the space $\mathcal{H}$. If $\mathcal{A}$ is any set in R_n, then $\mu(\mathcal{A})$ is the measure of this set.

1. As indicated above, any set $\mathcal{M} \subset R_n$ of positive measure permits one to define a special norm in the space $\mathcal{H}$, namely:

$$\|\hat{u}(\xi)\|_M = \left(\int_M |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}. \tag{3}$$

We shall be interested in those sets $\mathcal{M}$ for which

$$\|\hat{u}(\xi)\|_M \geq C(\mathcal{M}) \|\hat{u}(\xi)\|, \qquad C(\mathcal{M}) = \text{const} > 0. \tag{4}$$

Let $\mathfrak{M} = R_n \setminus \mathcal{M}$. It is obvious that (4) is satisfied if and only if there exists a constant $0 < \delta < 1$ such that

$$\|\hat{u}(\xi)\|_{\mathfrak{M}} \leqslant \delta \|\hat{u}(\xi)\| \tag{5}$$

for all $\hat{u}(\xi) \in \mathcal{H}$. We denote the exact lower bound of such δ by $\gamma(\mathfrak{M})$. It is easy to see that

$$\gamma(\mathfrak{M}) = \sup_E \left(\int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}. \tag{6}$$

Definition. A set $\mathfrak{M} \subset R_n$ is called a determining set if $\gamma(\mathfrak{M}) < 1$.

Theorem 1. *Every set $\mathfrak{M} \subset R_n$ of finite measure is determining.*

Proof. We first prove the theorem for a bounded set $\mathfrak{M}$. Let $\mathcal{M} = R_n \setminus \mathfrak{M}$. Suppose that the theorem is false. Then there exists a sequence $\{\hat{u}_n(\xi)\}_1^\infty \in E$ such that $\|\hat{u}_n(\xi)\|_M \to 0$ as $n \to \infty$. We now note that the unit sphere E, by Montel' s theorem, is compact in the space D of analytic functions with respect to the topology of this space. Therefore from the sequence $\{\hat{u}_n(\xi)\}_1^\infty$ one can extract a subsequence of analytic functions $\{\hat{u}_{n_k}(\xi)\}_{k=1}^\infty$ converging uniformly on every compact set. From the fact that $\|\hat{u}_n(\xi)\|_M \to 0$, it follows that the sequence $\{\hat{u}_{n_k}(\xi)\}_1^\infty$ converges uniformly to zero in the region between the ball containing the set $\mathfrak{M}$ and a larger ball; but then, by the maximum principle, it converges uniformly to 0 also on the set $\mathfrak{M}$. On the other hand, $\|\hat{u}_{n_k}(\xi)\|_{\mathfrak{M}} \to 1$ as $k \to \infty$. The contradiction proves the theorem.

To pass to the case of an unbounded set $\mathfrak{M}$, it suffices to use a lemma which itself turns out to be important in applications.

Lemma 1. *If $\mathcal{A} \subset R_n$ is an arbitrary set of finite measure, then*

$$\gamma(\mathcal{A}) \leqslant \mu(\mathcal{A}) \mu(\Omega).$$

* As usual, R^n denotes the n-dimensional space of real variables $x = (x^1, \ldots, x^n)$.

We now finish the proof of Theorem 1. Let the set $\mathfrak{M}$ be unbounded, but $\mu(\mathfrak{M}) < \infty$. If the theorem is false, then there exists a sequence of functions $\{\hat{u}_n(\xi)\}_1^\infty \in E$ such that, as $n \to \infty$, $\|\hat{u}_n(\xi)\|_{\mathfrak{M}} \to 1$. Split the set $\mathfrak{M}$ into two parts $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ so that $\mu(\mathfrak{M}_2) < 1/\mu(\Omega)$, while $\mathfrak{M}_1$ lies in some circle. Then from the sequence $\{\hat{u}_n(\xi)\}$ one can extract a subsequence $\{\hat{u}_{n'}(\xi)\}$ such that $\|\hat{u}_{n'}(\xi)\|_{\mathfrak{M}_1} \to 0$ and, consequently, $\|\hat{u}_{n'}(\xi)\|_{\mathfrak{M}_2} \to 1$. This, however, contradicts the fact that, according to Lemma 1, $\gamma(\mathfrak{M}_2) < 1$. Thus the proof of Theorem 1 is completely finished.

Theorem 2. *In the space R_n, any set $\mathfrak{M}$ of the form $\mathfrak{M} = R_{n-1} \times \mathfrak{B}$ is determining, where $\mathfrak{B}$ is a determining set on the line.*

The proof of the theorem follows immediately from Theorem 1. Let, for example, $\mathfrak{B}$ be a determining set on the axis ξ_1. Let $\hat{u}(\xi_1, \xi') \in \mathscr{H}$, where

$\xi' = (\xi_2, \dots, \xi_n)$. Then, for a fixed value of ξ', according to Theorem 1,

$$\int_{\mathfrak{B}} |\hat{u}(\xi_1, \xi')|^2 \, d\xi_1 \le \gamma(\xi') \int_{R_1} |\hat{u}(\xi_1, \xi')|^2 \, d\xi_1, \qquad \gamma(\xi') < 1, \tag{7}$$

and it remains to note that, since $\mathfrak{H} \ni \hat{u}(\xi_1, \xi')$, for fixed ξ', is again the Fourier transform of a function quadratically integrable in a bounded domain, the quantity $\gamma(\xi')$ does not depend on ξ', i.e. $\gamma(\xi') = \gamma < 1$. Integrating (7) with respect to ξ', we find

$$\int_{\mathfrak{M}} |\hat{u}(\xi)|^2 \, d\xi \le \gamma \int_{R_n} |\hat{u}(\xi)|^2 \, d\xi. \tag{8}$$

Thus the assertion of the theorem is proved.

2. In the preceding item we introduced the function $\gamma(\mathfrak{M})$, defined on the collection of all measurable sets $\mathfrak{M} \subset R_n$. The properties of this function, generally speaking, depend on the domain $\Omega \subset R^n$. Suppose first that Ω is a ball with center at the origin. Let $\mathfrak{A}_n$ be the set of all measurable sets $\mathfrak{M} \subset R_n$, and let $\mathfrak{B}_n \subset \mathfrak{A}_n$ be the collection of measurable sets of finite measure. Let $\mathfrak{U}_n$ be the group of unitary transformations of the space R_n.

Lemma 2. *The space $\mathfrak{H}$ is invariant with respect to transformations $u \in \mathfrak{U}_n$ and translations.*

Introduce in $\mathfrak{B}_n$ an order relation, considering that $\mathfrak{M}_2 > \mathfrak{M}_1$ if there is such a $u \in \mathfrak{U}_n$ that, after some translation, $u\mathfrak{M}_1 \subset \mathfrak{M}_2$ and $\mu(\mathfrak{M}_1) < \mu(\mathfrak{M}_2)$. With the aid of Lemma 2 the following theorem can be proved:

Theorem 3. *On the set $\mathfrak{B}_n$ the function $\gamma(\mathfrak{M})$ is strictly monotone, i.e. from $\mathfrak{M}_1 > \mathfrak{M}_2$ it follows that $\gamma(\mathfrak{M}_1) > \gamma(\mathfrak{M}_2)$.*

We note that for arbitrary sets from $\mathfrak{A}_n$ Theorem 3 is no longer true. If, for example, $\Pi_1 = \{\xi : 0 \le \xi_1 < \infty, \ |\xi_2| < 1\}$, $\Pi_2 = \{\xi : 1 \le \xi_1 < \infty, \ |\xi_2| < 1\}$, then it is easy to verify that $\gamma(\Pi_1) = \gamma(\Pi_2)$, although $\Pi_1 \supset \Pi_2$ and $\mu(\Pi_1 \setminus \Pi_2) > 0$. Nevertheless, Theorem 3 can be extended to a certain class of sets of infinite measure.

Theorem 4. *If $\mathfrak{M}_1 = N_1 \times R_p$, and $\mathfrak{M}_2 = N_2 \times R_p$, $p < n$, with N_1 and $N_2 \subset \mathfrak{B}_{n-p}$ and $N_1 > N_2$, then $\gamma(\mathfrak{M}_1) > \gamma(\mathfrak{M}_2)$.*

In the case of an arbitrary domain Ω, the space $\mathfrak{H}$ is no longer invariant with respect to all transformations $u \in \mathfrak{U}_n$, and therefore Theorems 3 and 4 are true only when the order relation in $\mathfrak{B}_n$ is introduced with the aid of the identity transformation. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are sets of one and the same measure, then, generally speaking, $\gamma(\mathfrak{M}_1) \neq \gamma(\mathfrak{M}_2)$. However, the following theorem is valid:

Theorem 5. *Let $\{\mathfrak{M}_\alpha\}$ be an arbitrary collection of sets with $\mu(\mathfrak{M}_\alpha) \leq \mu = \mathrm{const}$, the diameters d_α of which are bounded by one number d. Then there exists a number $\lambda < 1$ such that for all α, $\gamma(\mathfrak{M}_\alpha) < \lambda$.*

Of considerable interest is the formulation of a criterion that makes it possible to find all determining sets. Below we give a condition necessary in order that $\gamma(\mathfrak{M}) < 1$ hold. Let $\mathfrak{M}$ be an arbitrary measurable set in R_n. Denote by $\mathcal{R}_a^N$ the sphere of radius N with center at the point a. Let also $\mu_a^N(\mathfrak{M}) = \mu(\mathfrak{M} \cap \mathcal{R}_a^N)$ and

$$\rho_{\mathfrak{M}}(a, N) = \frac{\mu_a^N(\mathfrak{M})}{C_n N^n},$$

where C_n is the volume of the unit ball in R_n.

Definition. The number

$$\beta(\mathfrak{M}) = \lim_{N \to \infty} \sup_{a \in R_n} \rho_{\mathfrak{M}}(a, N)$$

will be called the **asymptotic density** of the set $\mathfrak{M}$.

It is not hard to see that $0 \leq \beta(\mathfrak{M}) \leq 1$, with $\beta(R_n) = 1$, and, if $\mathfrak{M}$ is a bounded set, then $\beta(\mathfrak{M}) = 0$. The set of black cells of an infi-

chessboard has asymptotic density equal to 1/2. In general, one can show that for any number $0 \leq \nu \leq 1$ there exists a set $\mathcal{N}_\nu$ with $\beta(\mathcal{N}_\nu) = \nu$.

Theorem 6. *If the asymptotic density of the set $\mathfrak{M}$, $\beta(\mathfrak{M}) = 1$, then $\gamma(\mathfrak{M}) = 1$.*

The question of the sufficiency of the condition given remains open.

3. In applications, the question of the equivalence of norms generated by two sets $\mathcal{M}$ and $\mathcal{N}$ turns out to be important. One can show that *if $\mathcal{M} \subset \mathcal{N}$ are bounded sets and $\mu(\mathcal{M}) < \mu(\mathcal{N})$, then the norms $\|\cdot\|_M$ and $\|\cdot\|_N$ are not equivalent.* Therefore the following theorem is of particular interest:

Theorem 7. *Let* Π *be a strip in the space* R_2*, and let* $\mathcal{M} \subset \Pi$ *be a set of finite measure. Then, if we denote* $\mathfrak{M} = \Pi \setminus \mathcal{M}$*, the norms* $\| \cdot \|_\Pi$ *and* $\| \cdot \|_{\mathfrak{M}}$ *turn out to be equivalent, i.e., there exists a constant* c *such that for all* $\hat{u}(\xi) \in \mathcal{H}$

$$\|\hat{u}\|_{\mathfrak{M}} \leq \|\hat{u}\|_\Pi \leq c\|\hat{u}\|_{\mathfrak{M}}. \tag{9}$$

The proof of this theorem requires no new ideas in comparison with the proofs of Theorems 1–5.

If Π is fixed, then the constant c in (9) depends on the set $\mathfrak{M}$ and, generally speaking, increases as $\mu(\mathcal{M})$ increases. One can show, however, that if k and m are arbitrary (fixed) numbers and $\mathcal{M}_\alpha \subset \Pi$ is an arbitrary collection of sets, each of which consists of no more than k connected subsets with diameter less than m, then the constants c_α in (9) for all $\mathfrak{M}_\alpha$ are bounded above by one constant (cf. Theorem 5). This assertion is naturally generalized to the case of a space of arbitrary number of dimensions and makes it possible to establish a broad class of determining sets. Namely, the following is true:

Theorem 8. *Suppose that in* R_n *there is a vector* $\vec{\zeta}$ *such that every straight line parallel to* $\vec{\zeta}$ *intersects the set* $\mathfrak{M}$ *in no more than* k *intervals of length* $\leq m$*. Then* $\gamma(\mathfrak{M}) < 1$.

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Note: Figure translations are in progress. See original paper for figures.