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Abstract

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ON THE PROBLEM OF SCALAR DIFFRACTION FOR AN ELLIPTIC CYLINDER AND AN ELLIPSOID OF REVOLUTION

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Let e be an eccentricity separated from unity, i.e., one for which the condition $e \leq e_0 < 1$ is satisfied. The solution of the problem of scalar diffraction for an elliptic cylinder or an ellipsoid of revolution with such an eccentricity behaves, from the point of view of its dependence on the geometric parameters, in exactly the same way as the solution of the same problem, respectively, for a circular cylinder or a sphere. However, this property is violated if, for fixed values of the major semiaxis a and of the length λ of the incident wave, $e \to 1$. We shall show how to construct a general asymptotic theory for large $2\pi a/\lambda$, which, first, gives the known results for an eccentricity separated from unity[1a]; second, gives new results when e is sufficiently close to 1, namely when

$$2\pi b^2/(\lambda a) \ll 1, \qquad 2\pi a/\lambda \gg 1, \tag{1}$$

and, third, describes in principle the behavior of the solution of the scalar diffraction problem for any positive e. For brevity we shall consider only the Dirichlet problem for an elliptic cylinder B, and for the case of an ellipsoid of revolution we shall confine ourselves merely to the formulation of some results.

Introduce elliptic coordinates (ξ, η), for which $x = c \operatorname{ch} \xi \cos \eta$, $y = c \operatorname{sh} \xi \sin \eta$, so that on the surface B, $\xi = \xi_0 = \operatorname{sech}^{-1} e$. Then, in the case of the Dirichlet problem, the Green' s function with respect to a line source with coordinates (Ξ, τ) and its normal derivative at $\xi = \xi_0$ have the form[1a]

$$u(\xi, \eta, \Xi, \tau) = \lim_{s \to 0+} \frac{1}{4\pi} \int_\Gamma \frac{\varphi_1(\xi, \nu) w_{-1}(\Xi, \nu)}{w_{-1}(\xi_0, \nu)} \widetilde{G}(\eta, \tau, -\nu) \, d\nu \qquad (\Xi > \xi > \xi_0), \tag{2}$$

$$\frac{\partial u(\xi_0, \eta, \Xi, \tau)}{\partial \xi} = \lim_{s \to 0+} \frac{-1}{2\pi i} \int_\Gamma \frac{w_{-1}(\Xi, \nu) \, \widetilde{G}(\eta, \tau, -\nu)}{w_{-1}(\xi_0, \nu)} \, d\nu,$$

where w_{-1} is the solution in $L_2(\xi_0, \infty)$ of the equation

$$\frac{d^2y}{d\xi^2} + (\gamma^2 \operatorname{sh}^2 \xi + \nu)y = 0 \qquad (\gamma = c(-is + 2\pi/\lambda)); \tag{3}$$

φ_1 is the solution of equation (3) satisfying the boundary condition at ξ_0; $\widetilde{G}$ is the angular resolvent of the Green' s function, and the path of integration Γ is a straight line that separates the poles of $\widetilde{G}(\nu)$ from the zeros of $w_{-1}(\nu)$.

To obtain convenient representations of the integrals (2) in the shadow region, they must be evaluated by means of residues determined by the zeros $\nu_n(\xi_0, \gamma)$ of the solution $w_{-1}(\xi_0, \nu, \gamma)$. The key to the solution of the problem lies in finding the functional behavior of these zeros. This has been achieved by applying the theory of asymptotic solutions of differential equations with transition points in the case when condition (1) is satisfied,

or $\xi_0 \gg \varepsilon > 0$. In the remaining cases it was not possible to find an analytic expression for the zeros; however, they can be found numerically.

If $|\nu|$ is small in comparison with $|\gamma^2 \operatorname{sh}^2 \xi_0|$, one may expect that the point $w_{-1}(\xi_0, \nu)$ lies on the exponential part of the curve of the solution $w_{-1}(\xi, \nu)$, so that such values of ν cannot be zeros ν_n. If $\xi_0 \gg \varepsilon > 0$, then, when $|\nu|$, increasing, becomes approximately equal to $-\gamma^2 \operatorname{sh}^2 \xi_0$, one may expect that the point $w_{-1}(\xi_0, \nu)$ will lie on the oscillating part of the curve $w_{-1}(\xi, \nu)$. Therefore one may suppose that there exist zeros ν_n of order γ^2. One may also expect that it will be possible to find the form of these zeros by using the asymptotic representation of $w_{-1}(\xi, \nu)$ near such a simple turning point, where $\operatorname{sh}^2 \xi = -\nu\gamma^{-2}$ and $\xi \gg \xi_0$. However, for $\xi_0 \to 0^+$ and for $|\gamma| > N$, where γ is fixed, these arguments are not applicable. In fact, when in the limit $\xi_0 = 0$, these zeros are of order γ instead of γ^2. Thus, in order to obtain the solution of the diffraction problem for all positive e, it is necessary to find a suitable approximation to the solution (3) in the region $k|\gamma| \ll |\nu| \ll K|\gamma|^2$. This asymptotic theory of the solution (3) will be briefly described below.

Let us write equation (3) in the form

$$\frac{d^2y}{d\xi^2} + \gamma^2(\operatorname{sh} \xi - \operatorname{sh}^2 \varepsilon)y = 0 \tag{4}$$

so that it can be compared with Weber' s equation $d^2v/dz^2 - \gamma^2(z^2 - \varepsilon^2)v = 0$, whose standard solutions are the functions $v_n(z, \gamma, \varepsilon) \equiv D_{1/2(\sigma\gamma\varepsilon^2-1)}(\sqrt{2\gamma}\, e^{1/2n\pi i} z)$, $\sigma = (-1)^n$, $\quad n = 0, \pm1, \dots$.

If z and ξ are connected by the relation

$$\int_\varepsilon^z (t^2 - \varepsilon^2)^{1/2}\, dt = \int_\varepsilon^\xi (\operatorname{sh}^2 t - \operatorname{sh}^2 \varepsilon)^{1/2}\, dt,$$

then the solutions v_n give an asymptotic representation of the solution of equation (4) for large $|\gamma|$. The zeros ν_n of the solution w_{-1} will approximately coincide with the zeros of the solution $v_{-1}[z(\xi_0), \gamma, \varepsilon]$, considered as a function of ν. After the functional form of the zeros ν_n has been determined, the remaining steps in constructing the residue series are carried out without particular difficulty.

If ε is separated from zero, then v_{-1} is well described in terms of Airy functions (2). This approximation leads to results (1a). If $\varepsilon^2 = O(\gamma^{-1})$, v_{-1} is a parabolic-cylinder function of bounded index. If this index is not too large, the zeros of the solution v_{-1} can be found by using the representation $D_\mu(x)$ by a power series (3a). (When $|\sqrt{2\gamma}\, z|$ is large, v_{-1} has no zeros. This follows directly from the standard representation of $D_\mu(x)$ for large $|x|$.) Thus, under condition (1) one can find that

$$\nu_n = \gamma\sigma_n \sim \left[(4n+3)i + \frac{(-1)^n 8 e^{\pi i/4}\gamma^{1/2}\xi_0}{\Gamma(-1/2-n)n!}\right] \quad (n = 0, 1, ...); \tag{5}$$

$$\frac{\partial w_{-1}(\xi_0, \nu_n)}{\partial \nu} \sim$$

$$\sim \frac{\pi^{1/2}}{4}\, 2^{(-i\sigma_n-1)/4}(-1)^n n!\, e^{(-\pi\sigma_n+5\pi i)/8}\gamma^{(i\sigma_n-5)/4}\left[1 + O(\gamma^{1/2}\xi_0) + O(\gamma^{-1})\right].$$

The asymptotic approximation of the angular resolvent of the Green' s function $\widetilde{G}(\eta, \tau, -\nu_n)$ is obtained by a direct application of the McKelvey theory (3a). In the case when ε^2 has a higher order than γ^{-1}, but is not separated from zero, it is not possible to find the functional form of the zeros either by representing v_{-1} by means of a power series, or by representing v_{-1} with the aid of Airy functions, when v_{-1} (considered as a parabolic-cylinder function) has large values of the index and argument.

Below we give two results obtained under conditions (1) (γ is taken for $s = 0$):

$$\frac{\partial u(\xi_0, \Xi, \eta, \tau)}{\partial \xi} \sim \sum_0^\infty \frac{4\cdot 2^{(i\sigma_n+1)/4}(-1)^{n+1}\gamma^{(5-i\sigma_n)/4}}{n!(\pi\gamma\ \mathrm{sh}\,\Xi)^{1/2}}\left(\frac{\mathrm{ch}\,\Xi - 1}{\mathrm{sh}\,\Xi}\right)^{-i\sigma_n/2} \times$$

$$\times \exp[-i\gamma(\mathrm{ch}\,\Xi - 1) + \pi(\sigma_n - 5i)/8]\, \widetilde{G}(\eta, \tau, -\nu_n).$$

Here

$$\widetilde{G}(\eta, \tau, -\nu_n) \sim \gamma^{(i\sigma_n-1)/2}\frac{e^{2i\gamma+(1-i\sigma)\pi i/4}}{4\pi}\left[\frac{4(1-\cos\tau)(1+\cos\eta)}{\sin^2\tau\,\sin^2\eta}\right]^{(-i\sigma_n+1)/4} \times$$

$$\times \exp[-i\gamma(2+\cos\eta-\cos\tau)]\left[\Gamma^2\left(\frac{1-i\sigma_n}{4}\right)-i\gamma\Gamma^2\left(\frac{3-i\sigma_n}{4}\right)\right],$$

when $\gamma(1-\cos\tau) \ll 1$ and $\gamma(1+\cos\eta) \ll 1$. For the case $\gamma(1-\cos\tau) \ll 1$ and $\gamma(1+\cos\eta) \gg 1$,

$$\widetilde{G}(\eta,\tau,-\nu_n) \sim \frac{1}{4\pi^{1/2}}\left[\frac{2(1+\cos\tau)}{\gamma\sin^2\tau}\right]^{(1-i\sigma_n)/4} e^{-i\gamma(1-\cos\tau)+\pi\sigma_n/8}\times$$

$$\times\left\{\frac{e^{3\pi i/8}}{\sqrt{\gamma\sin\eta}}\Gamma^2\left(\frac{1-i\sigma_n}{4}\right)[A_n+B_n]+\right.$$

$$\left.+2e^{-3\pi i/8}\left[\frac{1+\cos\eta}{\sin\eta}(1-\cos\tau)\right]^{1/2}\Gamma^2\left(\frac{3-i\sigma_n}{4}\right)\left[\frac{\Gamma\left(\frac{1-i\sigma_n}{4}\right)}{\Gamma\left(\frac{3-i\sigma_n}{4}\right)}A_n+B_n\right]\right\},$$

where

$$A_n = \frac{(\sin\eta)^{-i\sigma_n/2}e^{i\gamma(1-\cos\eta)}}{\Gamma\left(\frac{1-i\sigma_n}{4}\right)}, \qquad B_n = \frac{(\gamma\sin\eta)^{i\sigma_n/2}e^{i\gamma(3+\cos\eta)+\pi i/4}}{\Gamma\left(\frac{1+i\sigma_n}{4}\right)}.$$

These results were obtained in collaboration with Dr. R. F. Goodrich of the Radiation Laboratory of the University of Michigan.

Case of the ellipsoid of revolution. Let now ξ and η denote ellipsoidal coordinates corresponding to an ellipsoid of revolution. The role of equation (4) is played by the equation $w'' + \{\gamma^2[1+2\varepsilon(\xi^2-1)^{-1}] + (\xi^2-1)^{-2}\}w = 0$. This equation is compared with the equation $d^2W/dz^2+[\gamma^2(1+\varepsilon z^{-1})+(2z)^{-2}]W=0$, whose solutions are the Whittaker functions $W_{\varepsilon\gamma/2i,\,0}(2i\gamma z)$. The main problem is to find an asymptotic approximation of these functions uniformly for large $|\gamma|$ and for $0 \le |\varepsilon| \le \varepsilon_0$. For ε separated from zero, the study of this approximation gives the results (16). For small values of $\gamma(\xi_0-1)$, i.e. $\varepsilon = O(\gamma^{-1})$, the application of the Mac-Kelvy theory (36) gives results corresponding to the results obtained for the cylinder under condition (1). For intermediate values of ε the qualitative theory is again inapplicable. Below are placed some separate results for small values of $\gamma(\xi_0-1)$.

a) **Dirichlet problem.** If $|\gamma(1+\eta)| \gg 1$, then

$$\frac{\partial v(\xi_0,\eta,\Xi,1)}{\partial\xi} \sim \sum_0^\infty \frac{(-1)^n 2^{1/2} e^{i\gamma(\eta-\Xi)}(\xi_0+1)^{1/2}}{\ln[2i\gamma(\xi_0-1)](1-\eta^2)^{1/2}(\Xi^2-1)^{1/2}}\left(\frac{1+\eta}{1-\eta}\right)^{n+1/2}.$$

b) **Neumann problem.** If $|\gamma(1+\eta)| \ll 1$, then

$$v(\xi_0, \eta, \Xi, 1) \sim \sum_0^\infty -2^{-2n} i^n n! \gamma^{-n} e^{i\gamma(\eta-\Xi)} [(1+\xi_0)(1-\eta)(\Xi^2-1)]^{-1/2},$$

but if $|\gamma(1+\eta)| \gg 1$, then

$$z^1(\xi_0, \eta, \Xi, 1) \sim \sum_0^\infty (-1)^n 2^{1/2} e^{i\gamma(\eta-\Xi)} \left[((1-\eta^2)(1+\xi_0)(\Xi^2-1))\right]^{-1/2} \left(\frac{1+\eta}{1-\eta}\right)^{n+1/2}.$$

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Note: Figure translations are in progress. See original paper for figures.