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Abstract

Full Text

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ON INVERSE SPECTRA AND DIMENSION

(Presented by Academician P. S. Aleksandrov on 8 II 1961)

I. A spectrum $S = \{P_\alpha, \delta_\alpha^\beta\}$ (for the definition of a spectrum see in (^{1}) or in (^{2,3})) is called **polyhedral** if the spaces P_α are polyhedra given in some of their triangulations. If, for certain subdivisions P_β and P_α, the projections δ_α^β are simplicial and affine on the simplexes of the subdivision, then the spectrum S is called **simplicial**. A spectrum $\Sigma = \{K_\alpha, \delta_\alpha^\beta\}$ will be called **combinatorial** if the spaces K_α are finite complexes and the projections δ_α^β are transitive and single-valued. These spectra will be assumed Hausdorff and essential (^{2}).

Remark. The complexes K_α are not assumed to be complete, i.e., the faces of a simplex $t_\alpha \in K_\alpha$ need not belong to K_α.

The complexes K_α may be regarded as topological T_0-spaces (^{1}); in this case the projections δ_α^β are simply continuous mappings and the dimension $\dim K_\alpha$ is defined for K_α. The spectrum $S = \{P_\alpha, \delta_\alpha^\beta\}$, respectively $\Sigma = \{K_\alpha, \delta_\alpha^\beta\}$, will be called r-dimensional, and we shall write $\dim S \leq r$, respectively $\dim \Sigma \leq r$, if for every α

$$\dim P_\alpha \ (\dim K_\alpha) \leq r.$$

It is easy to see that for the limit space X of an r-dimensional polyhedral or combinatorial spectrum one has

$$\dim X \leq r.$$

One can prove:

Theorem 1. *In order that a spectrum*

$$\Omega = \{X_\alpha, \delta_\alpha^\beta\}$$

with projections "onto," where the X_α are bicompacts (respectively, an essential spectrum of finite complexes), should give in the limit a bicompact X of dimension not greater than r in the sense of dim, *it is necessary and sufficient that for every X_α and its covering*

$$\gamma = \{O_i\}, \quad i = 1, \dots, s,$$

there exist an X_β, $\beta \geq \alpha$, such that in it one can inscribe a covering of multiplicity not greater than $r + 1$ in the covering by the inverse images

$$(\delta_\alpha^\beta)^{-1} O_i.$$

We shall call a complex K_α a **complex of length** l and write

$$\operatorname{ind} K_\alpha \leq l,$$

if the maximal number of elements in any naturally ordered chain of its simplexes is not greater than $l + 1$.

Example. For a complex K consisting of a tetrahedron t^3, its two-dimensional face t^2, an edge t^1 which is a face of t^2, and a vertex t^0 of the edge t^1, we have

$$\dim K = 0, \qquad \operatorname{ind} K = 3.$$

We shall say that the dimension of the senior simplexes of a complex K_α does not exceed r if it can be realized so that the dimension of the senior simplexes of the realization does not exceed r. It is clear that in this case

$$\dim K \leq r \quad \text{and} \quad \operatorname{ind} K \leq r.$$

And, finally, a spectrum $\Sigma = \{K_\alpha, \delta_\alpha^\beta\}$ will be called a **spectrum of length** l ($\operatorname{ind} \Sigma \leq l$) if for every α we have

$$\operatorname{ind} K_\alpha \leq l.$$

It is clear that

$$\dim \Sigma \leq r \quad \text{and} \quad \operatorname{ind} \Sigma \leq r,$$

if for every α the dimension of the senior simplexes of K_α does not exceed r.

Theorem 2. *If for a combinatorial spectrum* Σ *we have* $\operatorname{ind} \Sigma \leq r$, *then for the limit space* X *one has*

$$\operatorname{Ind} X \leq r.$$

Theorem 3. *If a bicompact* X *is the limit of an* r*-dimensional simplicial spectrum* S *with projections "onto," then it is the limit of a combina-*

of the spectrum Σ with at most r-dimensional senior simplexes of the complexes of this spectrum, i.e. $\operatorname{ind} \Sigma \leqslant r$ and $\dim \Sigma \leqslant r$.

Further, one can construct a bicompactum L, which is the limit of a one-dimensional combinatorial spectrum of length 1, but is not the limit of any one-dimensional polyhedral spectrum. For this bicompactum $\dim L = \operatorname{ind} L = \operatorname{Ind} L = 1$, i.e. we have a strengthening of the result obtained in (3)*, where the absence of a one-dimensional polyhedral spectrum was derived only for a bicompactum one-dimensional in the sense of dim, and followed from the non-coincidence, for this bicompactum, of the dimensions dim and ind (Ind). From two copies of the bicompactum L (analogously to the way O. V. Lokutsievskii did this in (5)) one can construct a bicompactum M for which $\dim M = 1$, while $\operatorname{ind} M = 2$; but then, by Theorem 2, the bicompactum M cannot be the limit of any combinatorial spectrum of length 1, i.e. for combinatorial spectra of the given length the sum theorem is not true.

In (3) it was shown (this also follows from Theorems 2 and 3) that from the existence, for a bicompactum X, of an r-dimensional simplicial spectrum it follows that $\operatorname{Ind} X \leqslant r$. It turns out that, for every $m = 1, 2, \ldots, \infty$, one can construct a bicompactum $A(m)$ such that $\dim A(m) = \operatorname{ind} A(m) = \operatorname{Ind} A(m) = 1$, and $A(m)$ will be the limit of a one-dimensional combinatorial spectrum of length one, but for which a simplicial spectrum will be no less than m-dimensional.

Construction of L and $A(m)$. Denote on the line the point $(-1/2)$ by α; the interval $[0,1]$ by I_α; the Cantor perfect set, arranged in the usual way on $[0,1]$, by C_α, and the point $(3/2)$ by $\alpha+1$. Then the sets $\alpha \cup I_\alpha \cup \alpha+1$ and $\alpha \cup C_\alpha \cup \alpha+1$ are naturally ordered. In this case we shall say that I_α or C_α is placed between α and $\alpha+1$.

We construct L. Consider the transfinite numbers $\alpha \leqslant \omega_1$. Place intervals I_α between the pair of numbers α and $\alpha+1$, for every $\alpha \in W(\omega_1) = \{\alpha < \omega_1\}$. The topology in the set

$$K = \bigcup_{\alpha \leqslant \omega_1} \alpha \cup \bigcup_{\alpha < \omega_1} I_\alpha$$

is given by the transitive order generated by the order of the numbers α and the sets $\alpha \cup I_\alpha \cup \alpha+1$. Multiply the set K by the Cantor perfect set C, and then glue the set $C \times \omega_1$ into an interval, identifying pairwise the endpoints of the intervals adjacent to the set C (assuming it arranged in the usual way on the interval $[0,1]$). The construction of the bicompactum L is finished. As is clear, it is similar to that carried out in (5).

We construct the bicompactum $A(m)$ for $m = 2$. Consider all transfinite numbers $\alpha \leqslant \omega_{\tau_1}$ and $\beta \leqslant \omega_{\tau_2}$, where ω_{τ_i} is the first ordinal number of regular cardinality τ_i (i.e. ω_{τ_i} is not a limit point of any subset of $W(\omega_{\tau_i})$ of cardinality less than τ_i). Place Cantor perfect sets C_α between every pair of numbers α and $\alpha+1$. The topology of the set

$$C_{\tau_1} = \bigcup_{\alpha \leqslant \omega_{\tau_1}} \alpha \cup \bigcup_{\alpha < \omega_{\tau_1}} C_\alpha$$

is given, as before, by the transitive order generated by the order of the numbers α and the sets $\alpha \cup C_\alpha \cup \alpha+1$. Analogously we obtain

$$C_{\tau_2} = \bigcup_{\beta \leqslant \omega_{\tau_2}} \beta \cup \bigcup_{\beta < \omega_{\tau_2}} C_\beta.$$

Suppose $\tau_2 > \tau_1$. Multiply the sets C_{τ_1} and C_{τ_2}. In this product, on the sets $C_{\tau_1} \times \omega_{\tau_2}$ and $\omega_{\tau_1} \times C_{\tau_2}$, glue, as was already done above, the Cantor perfect sets $C_\alpha \times \omega_{\tau_2}$ and $\omega_{\tau_1} \times C_\beta$ (for all $\alpha < \omega_{\tau_1}$ and $\beta < \omega_{\tau_2}$) into the intervals I_α and I_β. The bicompactum obtained will be the required $A(2)$. $A(\infty)$ is obtained from the discrete sum of the spaces $A(m)$, $m = 1, 2, \ldots$, by bicompactifying with one point.

II. One may consider spectra $S = \{X_\alpha,\ \delta_\alpha^\beta\}$, where the X_α are Hausdorff spaces, imposing certain restrictions on the projections δ_α^β and on the dimension of X_α, and obtain estimates for the dimension of the limit space X.

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* This weaker result was also obtained by S. Mardešić in (4).

Theorem 4. If the projections ω_α^β do not lower ind of closed subsets X_β (i.e. $\operatorname{ind}\omega_\alpha^\beta F \geq \operatorname{ind} F$, where F is closed in X_β), then from $\operatorname{ind} X_\alpha \leq r$ for every α it follows that $\operatorname{ind} X \leq r$.

Corollary 1. If in Theorem 4 the set $\Phi \subset X$ is a bicompactum, then ω_α does not lower the dimension ind of the set Φ for every α.

Corollary 2. If closedness is required of ω_α, then for every closed $F \subset X$ we have $\operatorname{ind}\omega_\alpha F \geq \operatorname{ind} F$ for every α.

Corollary 3. In particular, if $\dim X = r$, the X_α are spaces with a countable base, $\dim X_\alpha \leq r$, and ω_α^β are closed and zero-dimensional, then $\operatorname{ind} X = \dim X$. Hence it is clear that if for a bicompactum X we have $\dim X = r < \operatorname{ind} X$, then in the spectrum $S = \{\Phi_\alpha, \omega_\alpha^\beta\}$ for X, where the Φ_α are r-dimensional compacta (such a spectrum always exists, see (4)), all projections ω_α^β cannot be zero-dimensional.

Theorem 5. Let the bicompactum X be the limit of such a spectrum $S = \{X_\alpha, \omega_\alpha^\beta\}$ of bicompacta X_α that ω_α^β do not lower Ind of closed subsets X_β. Then: a) from $\operatorname{Ind} X_\alpha \leq r$ for every α it follows that $\operatorname{Ind} X \leq r$; b) for every α, $\operatorname{Ind}\omega_\alpha\Phi \geq \operatorname{Ind}\Phi$, where Φ is closed in X.

Corollary. If in the spectrum S the spaces X_α are compacta for which $\dim X_\alpha \leq \dim X$ and the projections ω_α^β are zero-dimensional, then $\dim X = \operatorname{ind} X = \operatorname{Ind} X$.

Example. For the transfinite line A we have $\dim A = \operatorname{ind} A = \operatorname{Ind} A = 1$; for it there even exists a one-dimensional simplicial spectrum, but it is not the limit of any one-dimensional spectrum of compacta with zero-dimensional projections, since every continuous function on it is finally constant.

III. Using spectra, one can prove the following assertion.

Theorem 6. Let there be a locally bicompact group G, its closed subgroup H, and the space of left (right) cosets $X = G/H$. If $\operatorname{ind} H < \infty$ or $\operatorname{ind} X < \infty$, then $\dim X = \operatorname{ind} X = \operatorname{Ind} X = \operatorname{ind} G - \operatorname{ind} H$.

Remark. The coincidence of dimensions for locally bicompact groups was proved in (6). The equality $\operatorname{ind} X = \operatorname{ind} G - \operatorname{ind} H$, when $\operatorname{ind} X < \infty$*, follows from the results of Mostert (7).

It also turns out that Theorem 7 is valid, which is a strengthening of a result of A. Arhangel' skii (8):

Theorem 7. A homogeneous space $X = G/H$ for a locally bicompact group G decomposes into a discrete sum of finally compact spaces, i.e. it is strongly paracompact, and hence normal.

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Proof-correction note. One can even construct a snake-like bicompactum N for which $\dim N = \operatorname{ind} N = \operatorname{Ind} N = 1$, but which will not be the limit of any one-dimensional polyhedral spectrum, in particular of a spectrum of simple arcs. From two copies of the bicompactum N one can construct a snake-like bicompactum P for which $\operatorname{ind} P = 2$, i.e. the sum theorem for the dimension ind is false even for snake-like bicompacta.

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* This case includes the case of a finite-dimensional group G.

Note: Figure translations are in progress. See original paper for figures.

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