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Abstract

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On Compact Topological Spaces

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It is well known that compact spaces can be represented (the terminology of Bourbaki, ([3]), pp. 108-109, is used) by inverse spectra of finite discrete spaces (see, for example, ([1]), pp. 50 and 51). In the present note a construction is described which makes it possible to obtain arbitrary compact spaces as direct spectra of finite discrete spaces.

Let Ω be a directed set. To each $\alpha \in \Omega$ there is assigned a set M_α (the intersection $M_\alpha \cap M_\beta$, for $\alpha \neq \beta$, is regarded as empty). On the set $M = \bigcup M_\alpha$ a reflexive and symmetric relation Δ is given. If $x \in M_\alpha$, then α will be called the **index** of the element x and will be denoted by Ix. We shall say that $x \equiv y\ (\alpha)$ $(x, y \in M,\ \alpha \in \Omega)$ if there exists an element $z \in M$ such that $Iz \geq \alpha,\ x\Delta z$, and $y\Delta z$.

The aggregate $\mathfrak{S} = (\Omega, M, \Delta)$ will be called a **spectrum** if property C holds:

Property C. For every $\alpha \in \Omega$ there exists an index $\beta \in \Omega$ such that from the relations $x \equiv y\ (\beta)$, $y \equiv z\ (\beta)$, and $Iy > \beta$ it follows that $x \equiv z\ (\alpha)$ (this index β will be denoted by $\alpha/2$).

Remark 1. For every $\alpha \in \Omega$ there exists an index $\beta > \alpha$ (we shall denote it by $\alpha/4$) such that from the relations $x\Delta y$, $y \equiv z\ (\beta)$, $z\Delta u$, $Iy, Iz > \beta$, it follows that $x \equiv u\ (\alpha)$.

Indeed, since $x \equiv y\ (\beta)$ and $z \equiv u\ (\beta)$, one may take as $\alpha/4$ any index exceeding α, $\alpha/2$, and $(\alpha/2)/2$.

We shall call a **thread** such a nonempty subset S of the set M that:

B1. The intersection $S \cap M_\alpha$ is either empty or contains exactly one element s_α.

B2. If $\beta > \alpha$ and s_α exists, then s_β also exists.

B3. If s_α and s_β exist, then $s_\alpha \Delta s_\beta$.

We shall say that the threads S and T are close of order α (in notation $S \equiv T\ (\alpha)$) if $s_\beta \equiv t_\gamma\ (\alpha)$ for all β and γ for which s_β and t_γ exist.

From this and from Remark 1 it follows:

Remark 2. If $s_\beta \equiv t_\gamma\ (\alpha/4)$ for some $\beta, \gamma > \alpha/4$, then $S \equiv T\ (\alpha)$.

Theorem 1. *The relation $S \equiv T\ (\alpha)$ defines a uniform structure on the set of threads.*

For the proof it is enough to verify the validity of conditions 1°–4°, indicated in ([3]) on p. 151. The first three of them are obvious. To prove the last, choose $\beta > (\alpha/4)/2$ and suppose that $R \equiv S\ (\beta)$ and $S \equiv T\ (\beta)$. Then $r_\beta \equiv t_\beta\ (\alpha/4)$ and, in view of Remark 2, $R \equiv T\ (\alpha)$.

In view of ([3]), Ch. II, § 1 and 4, with the constructed uniform structure there is associated a separated uniform structure, and hence also a separated uniform space (([3]), p. 162, Proposition 1). This uniform pro-

space will be called the **limit of the spectrum** $\mathfrak{S}$ and denoted by $\mathfrak{A}(\mathfrak{S})$, or by $\mathfrak{A}(\Omega, M, \Delta)$. The elements of the space $\mathfrak{A}(\mathfrak{S})$ will be called **pencils**.

Theorem 2. *If all M_α are finite, then for the precompactness of the space $\mathfrak{A}(\mathfrak{S})$ it is sufficient that the following hold:*

Condition II. *For every $\alpha \in \Omega$ and every pencil $\mathfrak{f}$ there is a fan $S \in \mathfrak{f}$ such that s_α exists.*

Indeed, let $\alpha \in \Omega$ and $M_\alpha = \{x^1, \dots, x^n\}$. Denote by U^i the set of those pencils each of which contains a fan S for which $s_\alpha = x^i$. If $\mathfrak{f}, \mathfrak{t} \in U^i$, then for suitable fans $S \in \mathfrak{f}$ and $T \in \mathfrak{t}$ we shall have $s_\alpha = t_\alpha = x^i$. Applying B3, we see that $S \equiv T\ (\alpha)$, i.e., that $\mathfrak{f} = \mathfrak{t}\ (\alpha)$. Consequently, the set U^i is small of order α. In view of condition II, the system $U^1, \dots, U^n$ forms a covering of the space. It remains to apply Theorem 4 from ([3]) (p. 182).

Let now E be a uniform space and α a certain entourage ([3]), p. 159). A subset G of the space E will be called an **α-net** if for every $a \in E$ there is an element $b \in G$ such that $[a, b]\alpha$.* If Ω is a filter of entourages of the space E, then every set of the form $G = \bigcup G_\alpha$, where G_α is some fixed α-net, will be called a **net space** of the space E. It is clear that $\overline{G} = E$.**

We next consider a spectrum $\mathfrak{S} = (\Omega, M, \Delta)$. We shall call this spectrum **proper** if for every $x \in M$ there is a fan S containing x. Thus, in a proper spectrum, for every $x \in M$ one may choose a fan S containing x, and denote by $\varphi(x)$ the pencil containing the fan S. The mapping φ thus obtained from the set M into the space $\mathfrak{A}(\mathfrak{S})$ will be called **injective**. It is easy to verify that $\varphi(M)$ is a net space.

Taking Remark 2 into account, it is not difficult to verify the validity of the following properties:

1. If $x \equiv y\ (\alpha/4)$, $lx,\ ly > \alpha/4$, then $[\varphi(x), \varphi(y)]\alpha$.
2. If $[\varphi(x), \varphi(y)]\alpha$, then $x \equiv y\ (\alpha)$.

Theorem 3. *Every separable compact space is isomorphic to some $\mathfrak{A}(\Omega, M, \Delta)$, where all M_α are finite, while the spectrum itself is proper and satisfies condition*

II.

Proof. Let E be an arbitrary compact space. It may be regarded as uniform ([3]), p. 178, Theorem 2). Denote by Ω the collection of symmetric open entourages of the corresponding uniform structure. If $\alpha, \beta \in \Omega$, put $\alpha \geq \beta$ if from $[a, b]\alpha$ it follows that $[a, b]\beta$. It is easy to see that Ω becomes a directed set. Let M_α be an α-net of the space E. We may assume that all M_α are finite ([3]), p. 181, Theorem 3). Let $M = \bigcup M_\alpha$ (points coinciding in E, but belonging to different M_α, are regarded as distinct). If $x \in M_\alpha,\ y \in M_\beta$, then put $x\Delta y$ in the case when the intersection $\alpha(x) \cap \beta(y)$*** is nonempty.

(a) *For every $\alpha \in \Omega$ there exists $\beta \in \Omega$ such that for any two points $a, b \in E$ close of order β, there exists $x \in M_\alpha$ such that $a, b \in \alpha(x)$.*

Indeed, suppose that for every $\beta \in \Omega$ there are points a_β and b_β, close of order β, such that for every $x \in M_\alpha$ either $a_\beta \notin \alpha(x)$ or $b_\beta \notin \alpha(x)$. Let $A_\beta = \{a_\gamma;\ \gamma > \beta\}$. It is easy to understand that the system $\{A\}$ is centered. Let $c \in \bigcap A_\beta$. But $c \in \alpha(x)$ for some $x \in M_\alpha$. Find λ such that $\lambda(c) \subset \alpha(x)$. Let $\mu > \lambda$****. For some $\nu > \mu$

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* The notation $[a, b]\alpha$ means that a and b are close of order α.

** By $\overline{X}$ is denoted the closure of the set X.

*** $\lambda(c) = \{x;\ x \in E;\ x \equiv c(\lambda)\}$ (cf. ([3]), p. 158).

**** We shall say that $[a, b]\alpha \circ \beta$ if there exists a point c such that $[a, c]\alpha$ and $[c, b]\beta$; put ${}^2\alpha = \alpha \circ \alpha$ (cf. ([3]), p. 152).

find $a_\nu \in \mu(c)$. Then b_ν and c turn out to be close of order λ, i.e. $b_\nu \in \lambda(c) \subset \alpha(x)$. But $a_\nu \in \mu(c) \subset \lambda(c) \subset \alpha(x)$. A contradiction.

It is not hard to verify that as $\alpha/2$ one may take such a γ that $\gamma \overset{6}{>} \beta$, where β is the index found in assertion (a). Thus, $\mathfrak{S}(\Omega, M, \Delta)$ turns out to be a spectrum. It is easy to verify that this spectrum is proper.

Let $a \in E$. For each $\alpha \in \Omega$ choose $s_\alpha \in M_\alpha$ so that $a \in \alpha(s_\alpha)$. Then $S = \{s_\alpha\}$ turns out to be a fan. We shall denote the corresponding bunch by $\mathfrak{f}(a)$. If S is an arbitrary fan, put $S^\alpha = \{s_\beta;\ \beta > \alpha\}$. The system $\{S^\alpha\}$ turns out to be a base of a Cauchy filter in E. In view of Theorem 1 from ([3]) (p. 178), this filter has a limit $e(S)$. It is easy to see that $S \in \mathfrak{f}(e(S))$. This proves that the spectrum $\mathfrak{S}$ satisfies condition II. Hence it is also clear that the mapping $a \to \mathfrak{f}(a)$ is a mapping of E onto $\mathfrak{A}(\mathfrak{S})$. It is easy to verify that it is one-to-one. From assertion (a) it follows that it is uniformly continuous. Application of Theorem 2 and Corollary 2 from ([3]) (pp. 181 and 114) shows that the spaces E and $\mathfrak{A}(\mathfrak{S})$ are isomorphic.

Construction. Let a spectrum $\mathfrak{S} = (\Omega, M, \Delta)$ satisfy condition II, and let the spectrum $\mathfrak{S}' = (\Omega, M', \Delta')$ be proper. Suppose that a not necessarily single-valued mapping f of the set M into the set M' is given, with:

1. $Ix = Ix'$ for all $x' \in f(x)$.
2. For every $\alpha \in \Omega$ there exists an index $\beta \in \Omega$ such that $Ix, Iy > \beta$ and $x \equiv y(\beta)$ imply $x' \equiv y'(\alpha)$ for any $x' \in f(x)$ and $y' \in f(y)$.

Let, further, $\mathfrak{f}$ be a bunch from $\mathfrak{A}(\mathfrak{S})$ and $K' = \bigcup_{S \in \mathfrak{f}} f(S)$. If $x' \in K'$, then, in view of the properness of the spectrum $\mathfrak{S}'$, there exists a fan X' containing x'. We shall denote the corresponding bunch by $\mathfrak{x}'$. Put $\mathfrak{F}^\alpha = \{\mathfrak{x}';\ Ix' > \alpha\}$. Since $\mathfrak{S}$ satisfies condition II, $\{\mathfrak{F}^\alpha\}$ turns out to be a base of a filter. From 01 and 02 it is not hard to derive that this filter is a Cauchy filter. In view of ([3]) (p. 172, Theorem 2),

$$\lim \mathfrak{F}^\alpha = F(\mathfrak{f})$$

belongs to the completion $\overline{\mathfrak{A}(\mathfrak{S}')}$ of the space $\mathfrak{A}(\mathfrak{S}')$.

Remark 3. For every $\alpha \in \Omega$ there exists an index $\beta \in \Omega$ such that, for all $x \in M$ for which $Ix > \beta$, one has $[F(\mathfrak{f}), \varphi'(x')]\alpha$, where x' is any element of $f(x)$; $\mathfrak{f}$ is any bunch containing such a fan S that $x \in S$; φ' is an arbitrary injection mapping.

Indeed, let $\gamma \overset{2}{>} \alpha$ and $\beta > \gamma$ be such that $[F(\mathfrak{f}), \mathfrak{x}']\gamma$ for all $\mathfrak{x}' \in \mathfrak{F}^\beta$. If $Ix > \beta$, then, in view of 01, $Ix' > \beta$. Therefore $[\mathfrak{x}', \varphi'(x')]\beta$. Since $\mathfrak{x}' \in \mathfrak{F}^\beta$, it follows that $[F(\mathfrak{f}), \varphi'(x')]\alpha$.

Theorem 4. *The mapping F constructed in the construction described above is a uniformly continuous mapping of the space $\mathfrak{A}(\mathfrak{S})$ into $\overline{\mathfrak{A}(\mathfrak{S}')}$. Every uniformly continuous mapping Φ of the space $\mathfrak{A}(\mathfrak{S})$ into $\overline{\mathfrak{A}(\mathfrak{S}')}$ can be obtained in this way by using a single-valued mapping f.*

Proof. Let $\alpha \in \Omega$. Denote by β an index exceeding the indices indicated for $\alpha/4$ in Remark 3 and in condition 02. If $[\mathfrak{f}, \mathfrak{t}]\beta$, $S \in \mathfrak{f}$, $T \in \mathfrak{t}$, then for $\lambda, \mu > \beta$ we shall have $s_\lambda \equiv t_\mu(\beta)$, and hence $x' \equiv y'(\alpha/4)$ for any $x' \in f(s_\lambda)$, $y' \in f(t_\mu)$. Moreover, by Remark 3, $[F(\mathfrak{f}), \varphi'(x')]\alpha$ and $[F(\mathfrak{t}), \varphi'(y')]\alpha$. Taking $I1$ into account, we obtain $[F(\mathfrak{f}), F(\mathfrak{t})]\alpha$. This proves the uniform continuity of the mapping F.

Now suppose a mapping Φ is given. Denote by φ and φ' some injection mappings in the spectra $\mathfrak{S}$ and $\mathfrak{S}'$, respectively. If $x \in M$, then $\Phi(\varphi(x)) \in \overline{\mathfrak{A}(\mathfrak{S}')}$. Denote by $f(x)$ some element of M'_{Ix} satisfying the relation $[\varphi'(f(x)), \Phi(\varphi(x))]\overset{2}{I}x$. It is clear that condition 01 is fulfilled for the mapping f. Let, further, $\alpha \in \Omega$ and $\gamma \overset{5}{>} \alpha$. Choose-

choose $\beta > \gamma$ so that from $[a, b]\,\beta$ it follows that $[\Phi(a), \Phi(b)]\,\gamma$. If $Ix, Iy > \beta/4$ and $x \equiv y\,(\beta/4)$, then, by I1, $[\Phi(\varphi(x)), \Phi(\varphi(y))]\,\gamma$. Therefore $[\varphi'(x'), \varphi'(y')]\,\gamma$, and I2 gives $x' \equiv y'(\alpha)$. Thus the mapping f has property 02. Using the construction, we shall build the mapping F. Since $\varphi(\overline{M}) = \mathfrak{A}(\mathfrak{S})$, then, in view of ([3]) (p. 171, Theorem 1; p. 72, corollary), in order to prove the coincidence of the mappings F and Φ it is enough to establish that

$$F(\varphi(x)) = \Phi(\varphi(x))$$

for all $x \in M$. To verify this, take in the bundle $\varphi(x)$ some fan S. Let $\beta \in \Omega$ and let $\gamma > \beta$ be such that from $[a, b]\,\gamma$ there follow the relations $[F(a), F(b)]\,\beta$ and $[\Phi(a), \Phi(b)]\,\beta$. Suppose further that the index δ exceeds $\beta, \gamma/4$ and the index indicated for γ in Remark 3. From Remarks 2 and 3 it follows that $[\varphi(x), \varphi(s_\delta)]\,\gamma$ and $[F(\varphi)(s_\delta), \varphi'(f(s_\delta))]\,\gamma$. Hence, taking into account that, by the construction of the mapping f, one has

$$[\varphi'(f(s_\delta)), \Phi(\varphi(s_\delta))]\,\delta,$$

we obtain

$$[F(\varphi(x)), \Phi(\varphi(x))]\,\beta \circ \gamma \circ \delta \circ \beta.$$

Since $\beta \circ \gamma \circ \delta \circ \beta > \beta$, and β is arbitrary, everything is proved.

Theorem 5. Let $\mathfrak{S} = (\Omega, M, \Delta)$ be a regular spectrum satisfying condition Π, and let all M_α be abstract algebras (([2], p. 7) with one and the same system $\mathfrak{G}$ of operations; moreover, for each $g \in \mathfrak{G}$ one has

$$g(x_1, \dots, x_m)\,\Delta\, g(y_1, \dots, y_m),$$

if $x_i \Delta y_i$ for all i. Then the space $\overline{\mathfrak{A}(\mathfrak{S})}$ is an abstract algebra with the same system of operations.

For the proof it should be noted that every $g \in \mathfrak{G}$, as a mapping of

$$\underbrace{M \times \cdots \times M}_{m \text{ times}}$$

into M, satisfies conditions 01 and 02. After this it remains to apply Theorem 4.

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REFERENCES

[1] P. S. Aleksandrov, *UMN*, 2, No. 1 (17) (1947). [2] G. Birkhoff, *Theory of structures*, Moscow, 1952. [3] N. Bourbaki, *General Topology (Basic Structures)*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.