Doklady of the Academy of Sciences of the USSR

1961

SovietRxiv

Abstract

Full Text

Doklady of the Academy of Sciences of the USSR
1961, Vol. 141, No. 2

MATHEMATICS

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CONTINUAL ANALOGUES OF ORTHOGONAL POLYNOMIALS ON A SYSTEM OF INTERVALS

(Presented by Academician S. N. Bernstein, 12 VI 1961)

1. If a nondecreasing function $\sigma(\lambda)$ $(-\infty < \lambda < \infty)$ satisfies certain general conditions that were established and investigated in papers ([1−3]), then there is uniquely determined a family of functions $\varphi(x, \lambda)$, continuous in x $(x \geq 0)$, entire with respect to λ, possessing the property that, for any continuous finite function $f(x)$ $(x \geq 0)$, the integral

$$F(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)\, dx$$

belongs to $\mathcal{L}^2_\sigma$ and satisfies the relation

$$\int_{-\infty}^{\infty} |F(\lambda)|^2\, d\sigma(\lambda) = \int_0^\infty |f(x)|^2\, dx. \tag{1}$$

The function $\sigma(\lambda)$ may have prescribed intervals of constancy at a finite distance, but the general method of constructing the function $\varphi(x, \lambda)$ does not make it possible to trace how the presence of such "empty" intervals affects the analytic nature of the function $\varphi(x, \lambda)$ and of the "potential" $q(x)$ in the differential equation which, according to the general theory, the function $\varphi(x, \lambda)$ satisfies. Among the classical examples usually considered, there is likewise not a single one in which the continuous spectrum would fill the whole number axis or half-axis with a finite number of intervals removed. Such examples, undoubtedly interesting in themselves, could serve as a starting point for certain general constructions.

Since the function $\varphi(x, \lambda)$ is a continual analogue of a family of orthogonal polynomials, it seemed natural to me to investigate what might be obtained, in application to the continual case, from the method of papers ([4,5]), which made it possible to construct and study orthogonal polynomials on a system of finitely many intervals. The present paper is devoted to this investigation; here, however, I do not aim at the greatest generality allowed by the method mentioned.

2. Denote by E the positive half of the real axis from which the "empty" intervals

$$(\alpha_1, \beta_1),\ (\alpha_2, \beta_2), \dots, (\alpha_\rho, \beta_\rho) \qquad (0 < \alpha_1 < \beta_1 < \alpha_2 < \dots < \beta_\rho < \infty), \quad \text{(I)}$$

have been removed, and introduce the polynomials

$$R(\lambda) = \lambda(\lambda - \alpha_1)(\lambda - \beta_1) \cdots (\lambda - \beta_\rho), \qquad P(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2) \cdots (\lambda - \alpha_\rho).$$

We shall call the complex λ-plane cut along E the domain $\mathfrak{G}$. We agree that at points of the upper edge of the cut (β_ρ, ∞) the radical $\sqrt{R(\lambda)}$ has a positive value. Suppose that the spectral function $\sigma(\lambda)$ is absolutely continuous ($\sigma'(\lambda) = w(\lambda)$), and assume that $w(\lambda) = 0$ for $\lambda < 0$ and $\lambda \notin E$. We shall consider in parallel two cases, in which, respectively,

$$w(\lambda) \equiv w_C(\lambda) = \frac{P(\lambda)}{\pi\sqrt{R(\lambda)}}, \qquad w(\lambda) \equiv w_S(\lambda) = \frac{\sqrt{R(\lambda)}}{\pi P(\lambda)} \quad (\lambda \in E).$$

In the first case we shall denote the entire function $\varphi(x, \lambda)$ by $C(x, \lambda)$, and in the second by $S(x, \lambda)$. These are functions of normal type of order 1/2, which, as λ tends to infinity along the positive half of the real axis, have the following asymptotic representations:

$$C(x, \lambda) \sim \cos(x\sqrt{\lambda}), \qquad S(x, \lambda) \sim \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}.$$

We shall consider the function

$$\mathcal{E}(x, \lambda) = C(x, \lambda) + i\frac{\sqrt{R(\lambda)}}{P(\lambda)} S(x, \lambda),$$

which in the λ-plane is not only not entire, but not even single-valued. Therefore we introduce a Riemann surface, denote it by $\mathfrak{F}$, by gluing to the sheet $\mathfrak{G}$ a second similar sheet $\mathfrak{G}'$ in such a way that the transition lines of the surface $\mathfrak{F}$ are the intervals of the system (E). If ξ is a point of one of the sheets of the surface $\mathfrak{F}$, then the corresponding point of the other sheet will be denoted by ξ'.

On the surface $\mathfrak{F}$ the function $\mathcal{E}(x, \lambda)$ will already be single-valued. Considering together with it the function

$$\mathcal{E}(x, \lambda') = C(x, \lambda) - i\frac{\sqrt{R(\lambda)}}{P(\lambda)} S(x, \lambda),$$

it is not hard to see that the function $\mathcal{E}(x, \lambda)$ has at least one zero in each of the intervals of the system (1) (either on the upper or on the lower sheet of the surface $\mathfrak{F}$). Taking into account the behavior of the function $\mathcal{E}(x, \lambda)$ in a neighborhood of the infinitely distant point, we see that the function $\ln \mathcal{E}(x, \lambda)$ is an Abelian integral with a single, and moreover simple, pole at the point $\lambda = \infty$, while its logarithmic singular points with negative residues are only the points $\lambda = \alpha_1, \alpha_2, \dots, \alpha_\rho$. Therefore the number of zeros of the function $\mathcal{E}(x, \lambda)$ on the surface $\mathfrak{F}$ is equal to ρ, and hence all these zeros lie one in each of the intervals of the system (1); denote them by γ_k, so that $\alpha_k < \gamma_k < \beta_k$. These properties and the asymptotic formula

$$\mathcal{E}(x, \lambda) \sim e^{ix\sqrt{\lambda}} \tag{2}$$

at infinity completely determine the function $\mathcal{E}(x, \lambda)$, which also justifies its introduction. To construct the function $\mathcal{E}(x, \lambda)$ it is necessary to use the apparatus of the theory of hyperelliptic integrals.

3. For this purpose we introduce normal integrals of the first kind

$$\omega_k(\lambda) = \int_{\beta_\rho}^{\lambda} \frac{m_k(z)}{\sqrt{R(z)}}\,dz, \qquad m_k(z) = C_k z^{\rho-1} + \cdots,$$

which are determined by the conditions

$$\int_{\beta_{j-1}}^{\alpha_j} \frac{m_k(z)}{\sqrt{R(z)}}\,dz = \begin{cases} \frac{1}{2}\pi i & (j = k), \\ 0 & (j \neq k) \end{cases} \qquad (\beta_0 = 0;\ j, k = 1, 2, \dots, \rho).$$

Then we introduce an integral of the second kind with a pole at the point $\lambda = \infty$,

$$\omega(\lambda) = \int_{\beta_\rho}^{\lambda} \frac{M(z)}{2\sqrt{R(z)}}\,dz,$$

where the polynomial $M(z) = z^\rho + \cdots$ is determined by the conditions

$$\int_{\beta_{j-1}}^{\alpha_j} \frac{M(z)}{\sqrt{R(z)}}\,dz = 0 \qquad (j = 1, 2, \dots, \rho).$$

It is not difficult to verify that at infinity $\omega(\lambda) = \sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right)$.

Finally, let us introduce a normal integral of the third kind

$$\omega(\lambda;\gamma_k,\alpha_k)=\int_\infty^\lambda\left[\frac{\sqrt{R(z)}+\sqrt{R(\gamma_k)}}{z-\gamma_k}-\frac{\sqrt{R(z)}}{z-\alpha_k}+M_k(z)\right]\frac{dz}{2\sqrt{R(z)}},$$

where the polynomial $M_k(z)$ of degree $\rho-1$ is determined from the condition that the function $\exp\omega(\lambda;\gamma_k,\alpha_k)$ be single-valued in the domain $\mathfrak{F}$. In a neighborhood of the points $\lambda=\gamma_k$, $\lambda=\alpha_k$, this integral has the form

$$\omega(\lambda;\gamma_k,\alpha_k)=\ln(\lambda-\gamma_k)+\dots,\qquad \omega(\lambda;\gamma_k,\alpha_k)=-\frac{1}{2}\ln(\lambda-\alpha_k)+\dots.$$

It does not hurt to note that $\exp\omega(\lambda';\gamma_k,\alpha_k)=\exp\omega(\lambda;\gamma_k',\alpha_k)$.

With the aid of the Abelian integrals introduced, the function $\mathcal{E}(x,\lambda)$ may be represented in the form

$$\mathcal{E}(x,\lambda)=\exp\left\{ix\omega(\lambda)+\sum_{k=1}^{\rho}\omega(\lambda;\gamma_k,\alpha_k)\right\}. \qquad (3)$$

Indeed, its only zeros are the points γ_k, its only poles are the points α_k, and at ∞ the asymptotic equality (2) holds. In formula (3) we may also assign negative values to the parameter x.

It remains to determine to what problem of analysis the finding of the parameters $\gamma_1,\gamma_2,\dots,\gamma_\rho$, which of course depend on x, is reduced. This presents no difficulty. In fact, the function $\mathcal{E}(x,\lambda)$ must be single-valued on $\mathfrak{F}$. Therefore all moduli of periodicity of the Abelian integral

$$\ln\mathcal{E}(x,\lambda)=ix\omega(\lambda)+\sum_{k=1}^{\rho}\omega(\lambda;\gamma_k,\alpha_k)$$

must be equal to integral multiples of the number $2\pi i$. To find the moduli of periodicity, let us draw on $\mathfrak{F}$ the canonical cuts a_j,b_j $(j=1,2,\dots,\rho)$. As b_j we take a closed contour encircling, on the upper sheet, the cut (β_{j-1},α_j). The integrals taken along the contours b_j will be the moduli of periodicity A_j. From the construction of the function $\mathcal{E}(x,\lambda)$ it follows that all moduli A_j are equal to zero. As the cut a_j we take a closed contour which begins on the upper bank of the cut (β_{j-1},α_j), goes on the first sheet to the cut (β_ρ,∞), passes to the second sheet, and ends at the cut (β_{j-1},α_j) at its initial point. The integral taken along a_j will be the modulus of periodicity B_j. For $\omega(\lambda)$ this modulus is known and is equal to

$$\int_{\alpha_j}^{\beta_\rho} \frac{M(z)}{\sqrt{R(z)}}\, dz = -2U_j \qquad (j = 1, 2, \dots, \rho).$$

The corresponding modulus of periodicity for $\omega(\lambda; \gamma_k, \alpha_k)$, by a known theorem of the theory of Abelian integrals, is equal to $2\int_{\alpha_k}^{\gamma_k} d\omega_j(\lambda)$. Therefore our system of equations takes the form

$$-2ixU_j + \sum_{k=1}^{\rho} 2\int_{\alpha_k}^{\gamma_k} d\omega_j(\lambda) = 2n_j\pi i.$$

The numbers n_j are equal to zero, since the left-hand sides are real. We see that, in order to find the parameters γ_k, one has to solve the Jacobi problem of inversion of hyperelliptic integrals

$$\int_{\alpha_1}^{\gamma_1} d\omega_j(\lambda) + \int_{\alpha_2}^{\gamma_2} d\omega_j(\lambda) + \dots + \int_{\alpha_\rho}^{\gamma_\rho} d\omega_j(\lambda) = ixU_j \qquad (j = 1, 2, \dots, \rho),$$

where all the integrals are taken along segments of the real axis (on the sheet on which the corresponding point γ_k lies). It is clear from this that if the sign before x is changed, then all γ_k are replaced by γ_k'. Therefore the relation

$$\mathcal{E}(x, \lambda') = \mathcal{E}(-x, \lambda)$$

holds.

4. In our constructions we relied on the general theory of the inverse Sturm-Liouville problem. However, we could also have avoided relying on it, but then it would have been necessary to prove, in addition, that the functions $C(x, \lambda)$, $S(x, \lambda)$ generated by the function $\mathcal{E}(x, \lambda)$ satisfy the relations of continual orthogonality that are expressed by Parseval's equality (1). Such a direct proof is, of course, possible.

5. In the case of only one gap interval ($\rho = 1$), our constructions are greatly simplified and admit an interesting conclusion. We may, without loss of generality, put $\alpha_1 = k^2$ $(0 < k < 1)$ and $\beta_1 = 1$. By means of the conformal mapping

$$\lambda = \frac{1}{\operatorname{sh}^2(u; k)}, \qquad \frac{du}{d\lambda} = -\frac{1}{2\sqrt{\lambda(\lambda - k^2)(\lambda - 1)}}$$

the Riemann surface is mapped onto the rectangle of periods, the upper half of which corresponds to the first sheet, and the lower half to the second sheet of the surface. The function $\mathcal{E}(x,\lambda)$ is now represented by means of Jacobi theta-functions and has the form

$$\mathcal{E}(x,\lambda) = \frac{\theta_1(0)\,\theta_1(u-ix)}{\theta_1(u)\,\theta_1(ix)} e^{ixH'(u)/H(u)}. \tag{4}$$

To prove the representation (4), one must use the fact that $\mathcal{E}(x,\lambda)$ has a pole at the point $\lambda = k^2$, i.e. at $u = K + iK'$, has both periods $2K$, $2iK'$, and has the asymptotics indicated above as $\lambda \to \infty$.

It is not difficult to write down a differential equation whose solutions are the functions $C(x,\lambda)$, $S(x,\lambda)$. It has the form

$$-\frac{d^2y}{dx^2} + (1-k^2)\,\frac{\operatorname{cn}^2(x;k') - k^2\operatorname{sn}^2(x;k')}{\operatorname{cn}^2(x;k') + k^2\operatorname{sn}^2(x;k')}\,y = \lambda y, \tag{5}$$

where $k' = \sqrt{1-k^2}$. As we see, the potential $q(x)$ is a continuous function of x, having real period $2K'$. There exists a connection [6] between the continuous spectrum and the Lyapunov stability zones of the equation

$$-y'' + q(x)y = \lambda y \qquad (x > 0)$$

with periodic potential $q(x)$. On the basis of this connection one can directly prove that the spectrum of our equation (5) is formed by the intervals $[0,k^2]$, $[1,\infty]$. Let us also note that our equation (5) is directly connected with the classical Lamé equation, which in Jacobi functions has the form

$$y''(\xi) - n(n+1)k^2\operatorname{sn}^2(\xi;k)y(\xi) = \mu y(\xi),$$

where n and μ are constants. Putting in this equation $\xi = K + ix$, $n = 1$, $\mu = \lambda - 1 - k^2$, we obtain our equation (5).

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Received
9 VI 1961

REFERENCES

1. V. A. Marchenko, *Tr. Moskovsk. matem. obshch.*, **1**, 327 (1952).

2. I. M. Gel'fand, B. M. Levitan, *Izv. AN SSSR, ser. matem.*, **15**, 309 (1951).

3. M. G. Krein, *DAN*, **97**, No. 1 (1954).

4. N. I. Akhiezer, *DAN*, **134**, No. 1 (1960).

5. N. I. Akhiezer, Yu. Ya. Tomchuk, *DAN*, **138**, No. 4 (1961).

6. S. Wallach, *Am. J. Math.*, **70**, No. 4 (1948).

Note: Figure translations are in progress. See original paper for figures.