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1961

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Abstract

Full Text

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ON INTEGRAL THEOREMS FOR LARGE DEVIATIONS

(Presented by Academician A. N. Kolmogorov on 24 I 1961)

In paper ([1]) a number of new local limit theorems for large deviations were obtained. In the present paper we give the corresponding integral limit theorems. Throughout this note we shall use the notation of paper ([1]), namely, the notation of Section 1 of that paper for the case of identical distributions and the notation of Section 2 for the case of nonidentical distributions.

1. Let X_1, X_2 be a sequence of independent identically distributed random variables with finite variance $\sigma^2 > 0$ and expectation EX_1 equal to zero. Put

$$F_n(x) = \mathbf{P}\left\{\frac{X_1 + X_2 + \cdots + X_n}{\sigma\sqrt{n}} < x\right\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt. \quad (1)$$

Theorem 1. If

$$E \exp |X_1|^{\frac{4\alpha}{2\alpha+1}} < \infty \quad (2)$$

for some α ($0 < \alpha < 1/2$), then for $0 \le x \le n^{\alpha}/\rho(n)$, where $\rho(n)$ is an arbitrary function such that $\lim\limits_{n\to\infty} \rho(n) = +\infty$, as $n \to \infty$ we have

$$1 - F_n(x) = [1 - \Phi(x)] \exp\left\{\frac{x^3}{\sqrt{n}}\lambda^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right\}[1 + o(1)], \quad (3)$$

$$F_n(-x) = \Phi(-x) \exp\left\{-\frac{x^3}{\sqrt{n}}\lambda^{[s]}\left(-\frac{x}{\sqrt{n}}\right)\right\}[1 + o(1)] \quad (3a)$$

uniformly with respect to x. Here s is a nonnegative integer determined by the inequalities

$$\frac{s}{2(s+3)} < \alpha \le \frac{s+1}{2(s+3)}, \quad (4)$$

and $\lambda^{[s]}(t)$ is the segment of Cramér' s series consisting of its first s terms.

Condition (2) is necessary in order that relations (3) and (3a) hold for some integer $s \geq 0$, $0 \leq x \leq n^\alpha \rho(n)$, and $n \to \infty$ (even if not uniformly with respect to x), where α is some constant and $\rho(n)$ is some function such that $0 < \alpha < 1/2$, $\lim_{n\to\infty} \rho(n) = +\infty$.

The following very general assertion is also true:

Theorem 2. Suppose that for $0 \leq x \leq n^\alpha \rho(n)$ and all sufficiently large values of n the inequalities

$$1 - F_n(x) \leq c_0 e^{-c_1 x^2}, \qquad F_n(-x) \leq c_0 e^{-c_1 x^2},$$

hold, where $\rho(n)$ is some function satisfying the condition $\lim_{n\to\infty} \rho(n) = +\infty$; α, c_0, and c_1 are some positive constants, with $\alpha < 1/2$. Then condition (2) is satisfied for the given α.

From the assertions formulated one can obtain theorems concerning conditions for normal convergence (2). Let $\rho(n)$ denote a function increasing (arbitrarily slowly) to infinity, and let γ_m denote the cumulant (semi-invariant) of order m of the random variable X_1.

Corollary 1. Let $0 < \alpha < {}^1/_6$. Condition (2) is sufficient in order that, for $0 \leq x \leq n^\alpha / \rho(n)$ and $n \to \infty$, the relations

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_n(-x)}{\Phi(-x)} \to 1 \tag{5}$$

hold uniformly with respect to x, and is necessary in order that relations (5) hold for $0 \leq x \leq n^\alpha \rho(n)$ and $n \to \infty$.

Corollary 2. Let α satisfy condition (4) for some integer $s > 0$. Condition (2) and

$$\gamma_m = 0, \qquad (m = 3, 4, \dots, s + 2) \tag{6}$$

are sufficient in order that relations (5) hold for $0 \leq x \leq n^\alpha / \rho(n)$ and $n \to \infty$ uniformly with respect to x, and are necessary in order that (5) be fulfilled for $0 \leq x \leq n^\alpha \rho(n)$ and $n \to \infty$.

2. Consider a sequence of independent, generally speaking, not identically distributed random variables $X_1, X_2, \dots$ with mathematical expectations equal to zero. Denote by $F_n(x)$ the distribution function of the normalized sum of the random variables $X_1, X_2, \dots, X_n$.

Theorem 3. Let $0 < \alpha \leq {}^1/_6$; $\rho(n)$ be an arbitrary function satisfying the condition $\lim_{n\to\infty} \rho(n) = +\infty$. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E X_j^2 > 0, \qquad E \exp |X_j|^{\frac{4\alpha}{2\alpha+1}} \leq C \quad (j = 1, 2, \dots),$$

where C is some constant, then for $0 \le x \le n^{\alpha}/\rho(n)$ and $n \to \infty$ relations (5) hold uniformly with respect to x.

Theorem 4. If for the sequence of random variables under consideration the condition

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} EX_j^2 < \infty$$

is fulfilled, and if the distribution function $F_n(x)$ satisfies the conditions of Theorem 2, then

$$E \exp |X_j|^{\frac{4\alpha}{2\alpha+1}} < \infty$$

for all j $(j = 1, 2, ...)$.

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Received
8 I 1961

CITED LITERATURE

[1] V. V. Petrov, DAN, **134**, No. 3, 525 (1960). [2] Yu. V. Linnik, DAN, **133**, No. 6, 1291 (1960).

Note: Figure translations are in progress. See original paper for figures.