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Abstract

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A CRITERION FOR THE RELATEDNESS OF TWO BANACH SPACES

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In the paper ([1]) the notion of related Banach spaces was introduced. Two Banach spaces E_0 and E_1 are called **related** if there exists a family of Banach spaces $\{E_\alpha\}$ $(0 \leq \alpha \leq 1)$ with norms $\|x\|_\alpha$, possessing the following properties:

1) for $0 \leq \alpha < \beta \leq 1$, the space E_β is densely embedded in the space E_α, and

$$\|x\|_\alpha \leq \|x\|_\beta \quad (x \in E_\beta); \tag{1}$$

2) for $0 \leq \alpha < \beta < \gamma \leq 1$ and $x \in E_\gamma$,

$$\|x\|_\beta \leq \|x\|_\alpha^{\frac{\gamma-\beta}{\gamma-\alpha}} \|x\|_\gamma^{\frac{\beta-\alpha}{\gamma-\alpha}};$$

3) for $x \in E_1$,

$$\lim_{\alpha \to 1} \|x\|_\alpha = \|x\|_1.$$

In the present paper a criterion is given for two Banach spaces to be related.

We shall say that a Banach space E_1 is **normally embedded** in a Banach space E_0 if E_1 can be identified with some everywhere dense linear manifold in E_0, in such a way that $\|x\|_{E_0} \leq \|x\|_{E_1}$ $(x \in E_1)$.

Theorem 1. *Let the Banach space E_1 be normally embedded in the Banach space E_0. In order that the spaces E_0 and E_1 be related, it is necessary and sufficient that the unit ball S_1 of the space E_1 be a closed set in the topology induced in E_1 by the topology of the space E_0.*

Proof. Suppose that the spaces E_0 and E_1 are related, and let us show that the ball S_1 has the required property. If the contrary is assumed, then there will exist a sequence $x_n \in S_1$ and an element $x \in S_1$ $(\|x\|_1 = a > 1)$ such that $\|x_n - x\|_0 \to 0$. From 2) it follows that, for $0 < \alpha < 1$,

$$\|x_n - x\|_\alpha \leq \|x - x_n\|_0^{1-\alpha} \|x - x_n\|_1^\alpha \leq (a+1)^\alpha \|x - x_n\|_0^{1-\alpha} \to 0.$$

Thus the sequence x_n converges to x in all the spaces E_α ($\alpha < 1$). Using 3), one can choose α_0 so close to unity that $\|x\|_{\alpha_0} = a - \varepsilon > 1$. But $\|x_n\|_{\alpha_0} \leq \|x_n\|_1 \leq 1$, and, consequently, the sequence x_n cannot converge to x in the space E_{α_0}. We have arrived at a contradiction. The necessity is proved.

From the fact that the space E_1 is densely embedded in the space E_0, it follows that every linear functional $\langle x, x' \rangle$, bounded in the norm of E_0, is bounded also in the norm of E_1. Thus the space E_0' conjugate to E_0 is naturally regarded as a part of the space E_1'. It is not difficult to show,

that E_0' is dense in E_1' in the weak topology $\sigma(E_1', E_1)$. Moreover, from the inequality $\|x\|_{E_0} \leq \|x\|_{E_1}$ ($x \in E_1$) there follows the inequality $\|x'\|_{E_1'} \leq \|x'\|_{E_0'}$.

Introduce on the linear manifold E_1 a family of norms by the formula

$$\|x\|_\alpha = \sup_{x' \in E_0'} \frac{|\langle x, x' \rangle|}{\|x'\|_0^{1-\alpha} \|x'\|_1^\alpha}. \tag{2}$$

The quantity $\|x\|_\alpha$, obviously, has the properties of a norm. Further, for each $x \in E_1$ and $x' \in E_0'$, the function

$$\frac{|\langle x, x' \rangle|}{\|x'\|_0^{1-\alpha} \|x'\|_1^\alpha} = \frac{|\langle x, x' \rangle|}{\|x'\|_0} \left(\frac{\|x'\|_0}{\|x'\|_1} \right)^\alpha \tag{3}$$

is continuous and logarithmically convex in α on $[0, 1]$. Since $\|x'\|_1 \leq \|x'\|_0$ ($x' \in E_0'$), the function (3) increases monotonically in α. It is then easy to see that the supremum of all the functions (3) over $x' \in E_0'$, i.e. $\|x\|_\alpha$, is a continuous logarithmically convex function of α. Thus the set E_1, with the norms (2), forms a continuous incomplete normal scale of spaces with base E_1 (see (1)). The family of spaces E_α obtained by completing E_1 in the norms (2) will have properties 1), 2).

For $\alpha = 0$

$$\|x\|_0 = \sup_{x' \in E_0'} \frac{|\langle x, x' \rangle|}{\|x'\|_0^{1-\alpha} \|x'\|_1^\alpha} = \|x\|_{E_0},$$

and, by virtue of the density of E_1 in E_0, the completion of E_1 in the norm $\|x\|_0$ coincides with the space E_0.

If we show that

$$\|x\|_1 = \sup_{x' \in E_0'} \frac{|\langle x, x' \rangle|}{\|x'\|_0^{1-\alpha} \|x'\|_1^\alpha} = \|x\|_{E_1}, \tag{4}$$

then property 3) will hold for the family of spaces E_α, and the theorem will be proved.

Equality (4) is equivalent to the following:

$$\sup_{x' \in E'_0 \cap S'_1} \langle x, x' \rangle = \sup_{x' \in S'_1} \langle x, x' \rangle,$$

where S'_1 is the unit ball in the space E'_1. For the latter to hold it is necessary and sufficient that the regularly convex hull of the set $E'_0 \cap S'_1$, or, equivalently, its weak closure, coincide with S'_1 (see ([2])). In Bourbaki' s terminology (see ([3]), p. 275) this means that the characteristic of the set E'_0 in the space E'_1 must be equal to 1. For this it is necessary and sufficient (see ([3]), p. 275) that the ball S_1 be closed in the topology $\sigma(E_1, E'_0)$. But, by virtue of the convexity of the set S_1, its closure in the topology $\sigma(E_1, E'_0)$ will coincide with its closure in the topology induced by the topology of the space E_0 on E_1, and in this topology it is closed by assumption. The theorem is proved.

Corollary 1. If the space E_1 is normally embedded in the space E_0, and the conjugate space E'_0 is dense in the conjugate space E'_1, then the spaces E_0 and E_1, as well as the spaces E'_0 and E'_1, are related.

Corollary 2. A reflexive space E_1 is related to any Banach space in which it can be normally embedded.

Corollary 2 follows from Corollary 1. However, the fulfillment of the conditions of Theorem 1 in this case follows from one result of the paper ([4]).

Corollary 3. Let $E_0 \supset E_1 \supset E_2$ be three Banach spaces normally embedded in one another. If E_0 and E_2 are related, then E_1 and E_2 are also related.

Corollary 4. Let $E_0 \supset E_1 \supset E_2$ be three Banach spaces normally embedded in one another. If E_1 and E_2 are related and the unit sphere of E_2 is relatively compact in E_1, then E_1 and E_2 are related.

Corollary 5. Let E_0 be a reflexive Banach space, and let A be a linear operator with domain D_A dense in E_0, acting in E_0 and weakly closed. Introduce on D_A the norm $\|x\|_1 = \|x\|_0 + \|Ax\|_0$. In this norm D_A will be a Banach space E_1. The spaces E_0 and E_1 are related.

Let us consider some examples.

Let $C(0,1)$ be the space of all functions $x(t)$ continuous on $[0,1]$, with norm $\|x\|_0 = \max_{0 \le t \le 1} |x(t)|$, and let $C_1(0,1)$ be the space of all functions continuously differentiable on $[0,1]$, with norm $\|x\|_1 = \max_{0 \le t \le 1} |x(t)| + \max_{0 \le t \le 1} |x'(t)|$. It is not difficult to verify that for these spaces the conditions of Theorem 1 are satisfied and, consequently, they are related. Moreover, the unit sphere in $C_1(0,1)$ is relatively compact in $C(0,1)$. Therefore, by Corollary 4, the space $C_1(0,1)$ will be related to any space into which the space $C(0,1)$ can be normally embedded. For example, $C_1(0,1)$ is related to each of the spaces $L_p(0,1)$ $(1 \le p < \infty)$.

Similarly one can show that the space $L_p(0,1)$ and the space $C_\alpha(0,1)$, consisting of functions satisfying on $[0,1]$ the Hölder condition with exponent α, are related.

Using Corollary 5, one can show that the Sobolev space $W_p^{(l)}(D)$ with norm

$$\|x\|_1 = \left[\int_D |x(t)|^p \, dt\right]^{1/p} + \left\{\int_D \left[\sum_{k_1,k_2,\dots,k_l=1}^{\mu} \left(\frac{\partial^l x}{\partial t_{k_1} \cdots \partial t_{k_l}}\right)^2\right]^{p/2} dt\right\}^{1/p}$$

is related to the space $L_p(D)$. Applying the theorem on the complete continuity of the embedding operator $W_p^{(l)}(D)$ into $L_p(D)$, on the basis of Corollary 4 one may assert that the space $W_p^{(l)}(D)$ is related to all spaces $L_q(D)$ $(1 < q \leq p)$ under the corresponding normalization. On the other hand, by Corollary 3, the space $W_p^{(l)}(D)$ will be related to all spaces $L_q(D)$ and $W_q^{(m)}(D)$ for which the embedding theorems of S. L. Sobolev are valid.

If the spaces E_0 and E_1 are not related, then the question arises whether it is possible to introduce in E_1 an equivalent norm under which it becomes related to E_0.

Theorem 2. *Let the Banach space E_1 be normally embedded in the Banach space E_0. In order that an equivalent norm can be introduced in the space E_1, in which it will be related to the space E_0, it is necessary and sufficient that the intersection of the closure in the space E_0 of the unit ball S_1 of the space E_1 with the space E_1 be a bounded set in it.*

We give an example of two related spaces which, under an equivalent renorming, cease to be related. Take as E_0 the space $C(0,1)$, and as E_1 the space $C_1(0,1)$. These spaces, under the usual norm, are related. Introduce in E_1 the norm by the formula

$$\|x\|_1 = \max_{0\leq t\leq 1} |x(t)| \max_{0\leq t\leq 1} |x'(t)| + |x'(1)|.$$

It is easy to see that the unit ball $\|x\|_1 \leq 1$ is not closed in the sense of uniform convergence on $[0,1]$, and therefore the spaces E_0 and E_1 are not related.*

* The authors are indebted to M. A. Krasnosel' skii for this example.

For the construction of examples analogous to the one given, one may give the following general scheme. Let E_0 and E_1 be two normally embedded spaces, and let $\langle x, x' \rangle$ be a functional from the space E_1' not belonging to the space E_0'. Consider the hyperplane $\langle x, x' \rangle = a$ $(0 < a < 1)$, which is an everywhere dense linear manifold in the space E_0, and the balanced convex body $W = \{x;\ \|x\|_1 \leq 1,\ |\langle x, x' \rangle| \leq a\}$ of the space E_1. It may happen that W is not closed in the topology of the space E_0. Then the gauge function

$$\rho(x) = \max\left\{\|x\|_1,\ \frac{1}{a}|\langle x, x' \rangle|\right\},$$

defined by the body W, will define on E_1 a norm equivalent to the former one, while the space E_1, endowed with this norm, is not related to the space E_0.

Let us give an example of two normally embedded spaces which do not become related under any equivalent renorming of them. Denote by E_1 the Banach space consisting of double numerical sequences $x = (\xi_{nm})$ converging to zero, with norm

$$\|x\|_1 = \sup_{1 \leq n,m < \infty} |\xi_{nm}|.$$

Let $\{a_{nm}\}$ be a double sequence of positive numbers for which

$$\sum_{n,m=1}^{\infty} a_{nm} = 1.$$

Define the norm of x by the formula

$$\|x\|_0 = \sum_{n,m=1}^{\infty} a_{nm} \eta_{nm},$$

where

$$\eta_{nm} = \frac{|\xi_{nm} - n\xi_{n,m+1}|}{n+1}.$$

The norm $\|x\|_0$ has the property that the closure, in this norm, of the unit ball S_1 of the space E_1 is unbounded in the metric of the space E_1. By Theorem 2, the space E_0, obtained from E_1 by completion in the norm $\|x\|_0$, will not be related to E_1 if the space E_1 is endowed with any norm equivalent to the norm $\|x\|_1$.

In [1] it was shown that, for any two related spaces E_0 and E_1, one can construct a maximal continuous normal scale of spaces $\{G_\alpha\}$ $(0 \leq \alpha \leq 1)$. The space G_0 coincides with E_0, and if the spaces E_0 and E_1 are related, then G_1 coincides with E_1. If, however, E_0 and E_1 are not related, then the space E_1 is normally embedded in G_1. It follows from the preceding that the space G_1 can be constructed as follows: the unit sphere S_1 of the space E_1 is closed in the space E_0. The intersection of the closure $\overline{S}_1$ with the space E_1 is a convex body in E_1. The gauge function constructed from this body defines a new norm in E_1. The completion of E_1 in this norm gives the space G_1.

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Note: Figure translations are in progress. See original paper for figures.