THE CONNECTION BETWEEN TWO SPECTRAL FUNCTIONS CORRESPONDING TO ONE SECOND-ORDER DIFFERENTIAL EQUATION AND DIFFERENT INITIAL CONDITIONS

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Abstract

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MATHEMATICS

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THE CONNECTION BETWEEN TWO SPECTRAL FUNCTIONS CORRESPONDING TO ONE SECOND-ORDER DIFFERENTIAL EQUATION AND DIFFERENT INITIAL CONDITIONS

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1. Consider the Sturm–Liouville problem for the differential equation

$$Ly + \lambda y \equiv y'' + \{\lambda - q(x)\}y = 0, \tag{1}$$

where $q(x)$ is a function summable on every finite interval. Let α and β be arbitrary real numbers, and let $\varphi_\alpha(x, \lambda)$ be the solution of equation (1) satisfying the initial conditions

$$\varphi_\alpha(0, \lambda) = \cos\alpha, \qquad \varphi'_\alpha(0, \lambda) = \sin\alpha \tag{2}$$

and the boundary condition at $x = b$

$$\varphi_\alpha(b, \lambda)\sin\beta + \varphi'_\alpha(b, \lambda)\cos\beta = 0. \tag{3}$$

We shall consider the connection between the spectral functions $\rho_{\alpha,b}(\lambda)$ of the Sturm–Liouville problem corresponding to one and the same equation (1), the boundary condition (3), and the initial conditions (2) with different values of α. The results obtained will then be carried over to the half-infinite problem.

2. Suppose first that $\cos\alpha \neq 0$. Using the method set forth in [1], one can show that the function of the second kind

$$\psi_\alpha(x, z) = \int_{-\infty}^{\infty} \frac{\varphi_\alpha(x, \lambda)\, d\rho_{\alpha,b}(\lambda)}{z - \lambda}$$

satisfies the equation

$$L\psi_\alpha + z\psi_\alpha = 0 \tag{4}$$

and the conditions

$$y_\alpha(x,z)\psi'_\alpha(x,z) - y'_\alpha(x,z)\psi_\alpha(x,z) = 1 \tag{5}$$

($y_\alpha(x,z)$ is the solution of equation (4) with the initial conditions (2)), moreover

$$\psi_\alpha(x,z) = \lim_{n\to\infty} \psi_{\alpha,n}(x,z); \tag{6}$$

$$\psi'_\alpha(x,z) = \lim_{n\to\infty} \psi'_{\alpha,n}(x,z) \tag{7}$$

for $x > 0$. Here

$$\psi_{\alpha,n}(x,z) = \frac{1}{\cos\alpha}\int_{-\infty}^{\infty} \frac{\varphi_\alpha(x,\lambda)E_n(\lambda)}{z-\lambda}\,d\rho_{\alpha,\beta}(\lambda); \tag{8}$$

$$\psi'_{\alpha,n}(x,z) = \frac{1}{\cos\alpha}\int_{-\infty}^{\infty} \frac{\varphi'_\alpha(x,\lambda)E_n(\lambda)}{z-\lambda}\,d\rho_{\alpha,b}(\lambda); \tag{9}$$

$$E_n(\lambda) = \int_0^b f_n(x)\varphi_\alpha(x,\lambda)\,dx,$$

and the functions $f_n(x)$ satisfy the conditions: 1) $f_n(x) \geqslant 0$; 2) $f_n(x) = 0$ for $x \geqslant 1/n$; 3) $f_n(0) = f'_n(0) = 0$; 4) $f''_n(x)$ are continuous and 5) $\int_0^b f_n(x)\,dx = 1$.

For $x = b$, the functions $\psi_{\alpha,n}(x,z)$ satisfy the boundary conditions

$$\psi_{\alpha,n}(b,z)\sin\beta + \psi'_{\alpha,n}(b,z)\cos\beta =$$

$$= \int_{-\infty}^{\infty} \frac{[\varphi_\alpha(b,\lambda)\sin\beta + \varphi'_\alpha(b,\lambda)\cos\beta]\,E_n(\lambda)}{z-\lambda}\,d\rho_{\alpha,b}(\lambda) = 0$$

because $\rho_{\alpha,b}(\lambda)$ is a piecewise-constant function, and the values of λ at which the function $\rho_{\alpha,b}(\lambda)$ has a jump are eigenvalues of the problem.

Taking into account relations (6) and (7), for $x = b$ we obtain

$$\psi_\alpha(b,z)\sin\beta + \psi'_\alpha(b,z)\cos\beta = 0. \tag{10}$$

3. Let $\rho_{\gamma,b}(x_0,\lambda)$ be the spectral function corresponding to equation (1), the boundary conditions (3), and the initial conditions

$$\varphi_\gamma(x_0, \lambda) = \cos\gamma; \qquad \varphi'_\gamma(x_0, \lambda) = \sin\gamma \quad (0 \leqslant x_0 < b). \tag{11}$$

The function of the second kind

$$\psi_{\gamma,x_0}(x, z) = \int_{-\infty}^{\infty} \frac{\varphi_\gamma(x, \lambda)\, d\rho_{\gamma,b}(x_0, \lambda)}{z - \lambda}$$

satisfies equation (4) for $x > x_0$, as well as the boundary conditions (10). Therefore the functions $\psi_{\gamma,x_0}(x, z)$ and $\psi_\alpha(x, z)$ for $x \geqslant x_0$ are proportional:

$$\frac{\psi_{\gamma,x_0}(x, z)}{\psi_{\gamma,x_0}(x_0, z)} = \frac{\psi_\alpha(x, z)}{\psi_\alpha(x_0, z)}. \tag{12}$$

With the help of this relation, we write condition (5) for the functions $\psi_{\gamma,x_0}(x, z)$ and $\psi_\alpha(x, z)$ at $x = x_0$ in the following form:

$$y_\alpha(x_0, z)\psi'_\alpha(x_0, z) - y'_\alpha(x_0, z)\psi_\alpha(x_0, z) = 1,$$

$$\cos\gamma \cdot \psi'_\alpha(x_0, z) - \sin\gamma \cdot \psi_\alpha(x_0, z) = \frac{\psi_\alpha(x_0, z)}{\psi_{\gamma,x_0}(x_0, z)}. \tag{13}$$

Eliminating $\psi'_\alpha(x_0, z)$ from these equations, one can find the relation between the functions $\psi_{\gamma,x_0}(x_0, z)$ and $\psi_\alpha(x_0, z)$:

$$\psi_{\gamma,x_0}(x_0, z) = \frac{y_\alpha(x_0, z)\psi_\alpha(x_0, z)}{\cos\gamma + \psi_\alpha(x_0, z)\{\cos\gamma \cdot y'_\alpha(x_0, z) - \sin\gamma \cdot y_\alpha(x_0, z)\}}. \tag{14}$$

Since

$$\psi_{\gamma,x_0}(x, z) = y_{\gamma,x_0}(x, z) \int_{-\infty}^{\infty} \frac{d\rho_{\gamma,b}(x_0, \lambda)}{z - \lambda} + \frac{\theta_{x_0}(x, z)}{\cos\gamma}; \tag{15}$$

$$\psi_\alpha(x, z) = y_\alpha(x, z) \int_{-\infty}^{\infty} \frac{d\rho_{\alpha,b}(\lambda)}{z - \lambda} + \frac{\theta_0(x, z)}{\cos\alpha} \tag{16}$$

($y_{\gamma,x_0}(x, z)$, $y_\alpha(x, z)$, $\theta_{x_0}(x, z)$ are solutions of equation (4) with initial conditions (2) and (11) for $y_\alpha(x, z)$ and $y_{\gamma,x_0}(x, z)$, and with conditions $\theta_{x_0}(x_0, z) = 0$; $\theta'_{x_0}(x_0, z) = 1$ for $\theta_{x_0}(x, z)$), then with the aid of formula (14) one can find

Relation of functions

$$m_{\gamma,x_0}(b,z) = \int_{-\infty}^{\infty} \frac{d\rho_{\gamma,b}(x_0,\lambda)}{z-\lambda},$$

$$m_{\alpha}(b,z) = \int_{-\infty}^{\infty} \frac{d\rho_{\alpha,b}(\lambda)}{z-\lambda}.$$

The relation between the spectral functions $\rho_{\gamma,b}(x_0,\lambda)$ and $\rho_{\alpha,b}(\lambda)$ can be found from the relation between the functions $m_{\gamma,x_0}(b,z)$ and $m_{\alpha}(b,z)$ by means of the Stieltjes inversion formula:

$$\rho(\lambda_2) - \rho(\lambda_1) = \frac{1}{\pi} \lim_{\tau\to 0} \int_{\lambda_1}^{\lambda_2} [-\operatorname{Im} m(z)]\, d\sigma \tag{17}$$

$$\left(m(z) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{z-\lambda}; \qquad z = \sigma + \tau;\ \tau > 0 \right).$$

In particular, if we want to obtain the relation between the spectral functions corresponding to one and the same equation (1), to the boundary conditions (3), and to the initial conditions (2) with different values of α, then it is enough to put $x_0 = 0$ in formula (14). Finally we obtain

$$m_{\alpha_2}(b,z) = \frac{\cos^2\alpha_1 \cdot m_{\alpha_1}(b,z)}{\cos\alpha_2\left[m_{\alpha_1}(b,z)\cos\alpha_1 \sin(\alpha_1-\alpha_2) + \cos\alpha_2\right]}. \tag{18}$$

4. If one of the values $\cos\alpha_1, \cos\alpha_2$, for example $\cos\alpha_2$, is equal to zero, then the integral

$$\int_{-\infty}^{\infty} \frac{d\rho_{\alpha_2,b}(\lambda)}{z-\lambda}$$

does not converge. In this case one can find a relation between the functions

$$\overline{m}_{\alpha_2}(b,z) = \int_{-\infty}^{\infty} \frac{d\rho_{\alpha_2,b}(\lambda)}{(\lambda^2+1)(z-\lambda)}$$

and $m_{\alpha_1}(b,z)$. This relation is easily found from equations (13), for $x_0 = 0$, with the help of the equality

$$\int_{-\infty}^{\infty} \frac{\varphi_{\alpha}(x,\lambda)\, d\rho_{\alpha,b}(\lambda)}{(\lambda^2+1)(z-\lambda)} = \frac{-\frac{1+iz}{2}\psi_{\alpha}(x,-i) - \frac{1-iz}{2}\psi_{\alpha}(x,i) + \psi_{\alpha}(x,z)}{1-z}. \tag{19}$$

Finally we obtain:

$$\overline{m}_{\pi/2}(b,z) = \frac{1}{(1-z)\cos^2\alpha_1}\left\{\frac{1+iz}{2m_{\alpha_1}(b,-i)} + \frac{1-iz}{2m_{\alpha_1}(b,i)} - \frac{1}{m_{\alpha_1}(b,z)}\right\}. \quad (20)$$

The integral defining the function $\overline{m}_{\pi/2}(b,z)$ is meaningful.

5. If the half-line problem is considered, and $\rho_\alpha(\lambda)$, $\rho_\gamma(x_0,\lambda)$ are two spectral functions corresponding to equation (1), and also to the initial conditions (2) and (11), then

$$\rho_\alpha(\lambda) = \lim_{b\to\infty}\rho_{\alpha,b}(\lambda), \qquad \rho_\gamma(x_0,\lambda) = \lim_{b\to\infty}\rho_{\gamma,b}(x_0,\lambda).$$

Since the integrals

$$m_\alpha(b,z) = \int_{-\infty}^{\infty}\frac{d\rho_{\alpha,b}(\lambda)}{z-\lambda}, \qquad m_{\gamma,x_0}(b,z) = \int_{-\infty}^{\infty}\frac{d\rho_{\gamma,b}(x_0,\lambda)}{z-\lambda}$$

converge uniformly in b, which follows from the asymptotic behavior of the functions $\rho_{\alpha,b}(\lambda)$ and $\rho_{\gamma,b}(x_0,\lambda)$ as $|\lambda|\to\infty$ (see ([2])), then, by Helly' s theorem,

$$m_\alpha(z) = \int_{-\infty}^{\infty}\frac{d\rho_\alpha(\lambda)}{z-\lambda} = \lim_{b\to\infty} m_\alpha(b,z), \qquad m_{\gamma,x_0}(z) = \int_{-\infty}^{\infty}\frac{d\rho_\gamma(x_0,\lambda)}{z-\lambda} = \lim_{b\to\infty} m_{\gamma,x_0}(b,z).$$

Therefore, everything considered above is also valid for the semi-infinite interval upon replacing $\rho_{\alpha,b}(\lambda)$ and $\rho_{\gamma,b}(x_0,\lambda)$ by $\rho_\alpha(\lambda)$ and $\rho_\gamma(x_0,\lambda)$, and $m_\alpha(b,z)$ and $m_{\gamma,x_0}(b,z)$ by $m_\alpha(z)$ and $m_{\gamma,x_0}(z)$.

We note that, for the first time, the relation between the functions $m_\alpha(z)$ and $m_{\gamma,x_0}(z)$ for the special case was established in the paper ([3]).

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Note: Figure translations are in progress. See original paper for figures.