MATHEMATICS

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Abstract

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ESTIMATES OF TRIGONOMETRIC SUMS OF A SPECIAL TYPE AND THEIR APPLICATIONS

(Presented by Academician I. M. Vinogradov, 29 X 1960)

Consider the trigonometric sum

$$S=\sum_{x=1}^{N} e^{2\pi i\left(\frac{a_1 x}{p^n}+\frac{a_2 x^2}{p^{n-1}}+\cdots+\frac{a_n^n x}{p}\right)}, \tag{1}$$

where $(a_\nu, p)=1,\ \nu=1,2,\dots,n$.

Theorem 1. Let S be a sum of the form (1), $p \le N \le p^n$, $\log p \gg n^2 \log^3 n$. Then the inequality

$$|S| \le c_1 N^{1-\frac{c_2}{n^2}},$$

holds, where c_1, c_2 are absolute constants.

In the proof of this theorem one uses the theorem on the mean of I. M. Vinogradov (1), Lemma 1 of the paper (2), and the special form of the sum (1) (we make essential use of several coefficients of the polynomial

$$f(x)=\frac{a_1 x}{p^n}+\frac{a_2 x^2}{p^{n-1}}+\cdots+\frac{a_n^n x}{p}.$$

Trigonometric sums of the form (1) occur in the theory of Dirichlet L-series and in certain other number-theoretic questions.

From Theorem 1 and results of A. G. Postnikov (3) on Dirichlet L-series, the following theorem follows easily:

Theorem 2. Let $\chi(k)$ be a primitive character modulo $D=p^n$, p a prime >2, and $\log p \gg n^2 \log^3 n$.

Then, denoting by S_N the sum $\sum_{k=1}^{N}\chi(k)$, we have

$$|S_N| \le \begin{cases} p^2, & \text{if } N \le p^2, \\ c_3 N^{1-\frac{c_4}{n^2}}, & \text{if } p^2 \le N \le p^n, \end{cases}$$

where c_3, c_4 are absolute constants.

Further, in the usual way we obtain the theorem:

Theorem 3. Let $n^2 \log^3 n \ll \log p \le n^\theta$, $\chi(k)$ be a primitive character modulo $D = p^n$, p a prime > 2. Then $L(s, \chi)$ has no zeros in the region

$$|s| < c_5, \qquad \sigma > 1 - \frac{1}{\log^{\theta+1} D},$$

where c_5 is a constant.

Corollary. Let ε be an arbitrarily small positive quantity and let

$$n^2 \log^3 n \ll \log p \ll n^{2+\varepsilon}.$$

Then $L(s, \chi)$ has no zeros in the domain

$$|s| < c_5, \qquad \sigma > 1 - \frac{1}{\log^{1/3+\varepsilon} D}.$$

The best result previously obtained in the theory of zeros of Dirichlet L-series is the following (see ([4])): if ε is an arbitrarily small positive quantity and

$$\log p \ll \frac{n^\varepsilon}{\log^{3/4} n},$$

then $L(s, \chi)$ has no zeros in the domain

$$|s| < c_5, \qquad \sigma > 1 - \frac{1}{\log^{3/4+\varepsilon} D}.$$

Theorem 2 also improves the corresponding result of ([3]).

I express my gratitude to my adviser N. M. Korobov for the great assistance he gave me in carrying out this work.

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Note: Figure translations are in progress. See original paper for figures.