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Abstract

Full Text

MATHEMATICS

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ON CROSSED PRODUCTS OF A SEMIGROUP AND A RING

(Presented by Academician P. S. Aleksandrov on 2 XII 1960)

The notion of a crossed product, introduced by E. Noether, is widely used in the study of central simple algebras. In the book of N. Jacobson ([1]) crossed products of finite groups and fields are defined, and A. I. Tikhomirov ([2]) considered crossed products of a field and a certain semigroup of isomorphisms of this field. In the present note crossed products of an arbitrary associative ring with identity and an arbitrary semigroup with identity are introduced and studied. A group ring over a ring with identity is a special case of the crossed product of a group and a ring and, as it turns out, many properties of group rings remain valid for arbitrary crossed products.

Let G be an arbitrary semigroup with identity, and let K be an arbitrary associative ring with identity. Suppose there are given a single-valued mapping σ of the semigroup G into the group of automorphisms of the ring K and a family $\rho = \{\rho_{g,h}\}$ $(g, h \in G)$ of invertible elements of the ring K, satisfying the relations:

$$1) \qquad \rho_{g_1,g_2g_3} \cdot \rho_{g_2,g_3} = \rho_{g_1g_2,g_3} \cdot \rho_{g_1,g_2}^{g_3\sigma};$$

$$2) \qquad \alpha^{g_1\sigma \cdot g_2\sigma} = \rho_{g_1,g_2}^{-1} \cdot \alpha^{(g_1g_2)\sigma} \cdot \rho_{g_1,g_2}$$

for all $\alpha \in K$, where $g_1, g_2, g_3 \in G$. The family ρ is called a **factor system**.

To each element $g \in G$ we assign a symbol t_g and consider the set of all possible sums of the form

$$\sum_{g\in G} t_g\alpha_g \qquad (\alpha_g \in K),$$

in which only a finite number of the coefficients α_g are nonzero.

$$\sum_{g\in G} t_g\alpha_g = \sum_{g\in G} t_g\beta_g$$

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if and only if $\alpha_g = \beta_g$ for all $g \in G$. This set becomes an associative ring if the operations of addition and multiplication are defined as follows:

$$1) \quad \sum_{g\in G} t_g\alpha_g + \sum_{g\in G} t_g\beta_g = \sum_{g\in G} t_g(\alpha_g + \beta_g);$$

$$2) \quad t_g t_h = t_{gh}\rho_{g,h} \quad (g, h \in G);$$

$$3) \quad \alpha t_g = t_g \alpha^{g\sigma} \quad (\alpha \in K),$$

and for arbitrary elements the product is defined on the basis of the distributive law. We shall call this ring the **crossed product** of the semigroup G and the ring K with respect to the factor system ρ and the mapping σ, and denote it by (G, K, ρ, σ). Obviously, the element

$t_1\rho_{1,1}^{-1}$ is the identity of the ring (G, K, ρ, σ), and if the element g is invertible in the semigroup G, then $t_g^{-1} = (\rho_{1,1}\rho_{g^{-1},g})^{-1} t_{g^{-1}}$.

If the factor system ρ is the identity, i.e. $\rho_{g,h} = 1$ for all $g, h \in G$, and if σ maps the semigroup G to the identity automorphism of the ring K, then the crossed product is called the **semigroup ring** of the semigroup G over the ring K and is denoted by $R(G, K)$. If the semigroup is a group, then the semigroup ring is called a **group ring**.

Kaplansky ([3]) posed several problems on group rings. These problems were subsequently considered by a number of authors. In fact, these problems can be formulated for the much more general case of crossed products. In the present note a number of properties of crossed products are reported, in particular those related to the indicated problems.

We shall say that the crossed product (G, K, ρ, σ) of a semigroup G and a ring K with identity contains only **trivial divisors of the identity** if all divisors of the identity in (G, K, ρ, σ) have the form $t_g\varepsilon$, where g is an invertible element of the semigroup G with identity and ε is a divisor of the identity of the ring K. Elements of the indicated form will in fact be divisors of the identity.

A semigroup G is called **right-orderable** if a linear order is introduced in it, subject to the condition: $a < b$ implies $ax < bx$ for all $x \in G$.

Theorem 1. *If G is a right-orderable semigroup with cancellation and with identity, and K is an arbitrary associative ring without zero divisors with identity, then any crossed product (G, K, ρ, σ): 1) is a ring without zero divisors; 2) contains only trivial divisors of the identity; 3) is semiprimitive in the sense of Jacobson.*

As M. I. Zaitseva ([4]) showed, an RN-group with torsion-free factors ([5]) can be right-orderable. Consequently, Theorem 1 is a generalization of the theorem on

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group rings of RN-groups with torsion-free factors obtained by the author ([16]) by other methods. The question of the semiprimitivity of group rings was also considered by Villamayor ([6]) and Amitsur ([7,8]).

Theorem 2. *Let D be an arbitrary division ring and G an ordered group. Then every crossed product (G, D, ρ, σ) can be embedded in a division ring.*

Theorem 2 is a generalization of the Mal' tsev ([9])–Neumann ([10]) theorem on embedding the group algebra of an ordered group in a division algebra. In the proof A. I. Mal' tsev' s method is used.

Crossed products $(G_1, K, \rho_1, \sigma_1)$ and $(G_2, K, \rho_2, \sigma_2)$ are called **isomorphic** if there exists an isomorphism φ of the ring $(G_1, K, \rho_1, \sigma_1)$ onto the ring $(G_2, K, \rho_2, \sigma_2)$ such that

$$\varphi(x\alpha) = \varphi(x)\alpha, \quad \text{where } x \in (G_1, K, \rho_1, \sigma_1),\ \alpha \in K.$$

Theorem 3. *Let G_1 and G_2 be right-orderable groups, and K an arbitrary associative ring without zero divisors with identity. If the rings $(G_1, K, \rho_1, \sigma_1)$ and $(G_2, K, \rho_2, \sigma_2)$ are isomorphic, then the groups G_1 and G_2 are isomorphic in the group sense.*

Theorem 3 is a generalization of a theorem of S. D. Berman ([11]) on group rings of abelian groups without torsion, and of a theorem proved by the author that the group ring $R(G, K)$ of an RN-group with torsion-free factors over a ring K with the indicated properties determines the group G uniquely up to isomorphism.

A ring K is **regular in the sense of J. von Neumann** if for every $a \in K$ there exists an $x \in K$ such that $axa = a$. (For the basic properties of regular rings see, for example, ([12]).)

Theorem 4. *If G is a locally finite group and K is a regular ring in the sense of J. von Neumann in which one can uniquely divide by po-*

of any element of the group G, then any crossed product (G, K, ρ, σ) is also regular and, consequently, is semisimple in the sense of Jacobson.

In the case of group rings, Theorem 4 implies Auslander' s theorem [(13)], proved by methods of homological algebra. Other proofs in this case were given by McLaughlin [(14)] and Villamayor [(15)].

The converse theorem has been obtained only for crossed products of a special kind.

Theorem 5. *Let G be an arbitrary group and K an arbitrary associative ring with identity. If the crossed product $(G, K, 1, \sigma)$ with a unit system of factors is regular in the sense of Neumann, then the group G is locally finite and the ring K is regular.*

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Theorem 5 is a generalization of Villamayor' s theorem (15) and is proved by his method. In the case of regular group rings one can also show (13–15) that in the ring K one can divide uniquely by the order of any element of the group G.

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