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Abstract

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ON TRANSLATION SURFACES OF CONSTANT CURVATURE

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The aim of the present note is to prove the following propositions.

Theorem 1. *A translation surface of constant curvature is a cylinder or a plane.*

Theorem 2. *In elliptic space, a surface of constant curvature K is simultaneously a sliding surface only when $K = 0$.*

Theorem 3. *On a surface of constant curvature $K \neq 0$ there exists no Voss net.*

Proof. Refer the translation surface of constant curvature to the conjugate Chebyshev net of translation curves. The coefficients of the differential forms of the surface satisfy the conditions

$$E = G = 1, \qquad F = \cos\omega, \qquad M = 0. \tag{1}$$

From the Gauss equations

$$LN = K\sin^2\omega, \tag{2}$$

$$\omega_{uv} + K\sin\omega = 0 \tag{3}$$

and the Peterson–Codazzi equations

$$\sin\omega\, L_v = N\omega_u, \qquad \sin\omega\, N_u = L\omega_v. \tag{4}$$

it follows that

$$L^2 + \omega_u^2 = {U'}^2, \qquad N^2 + \omega_v^2 = {V'}^2; \tag{5}$$

$$({U'}^2 - \omega_u^2)({V'}^2 - \omega_v^2) = K^2\sin^4\omega. \tag{6}$$

U' and V' are the curvatures of the coordinate lines. If at least one of these functions is zero, the translation surface is cylindrical. Excluding this case, we take U, V as new independent variables. Excluding also the plane, we may assume $K \neq 0$.

Denote the partial derivatives of the function ω with respect to the variables U, V by p, q, r, s, t. In the new variables equations (3), (6) take the form

$$sU'V' + K \sin\omega = 0, \tag{7}$$

$$\varepsilon\sqrt{1-p^2}\sqrt{1-q^2} = s\sin\omega, \qquad \varepsilon^2 = 1.$$

Eliminating the functions U', V' from system (7) leads to the system

$$s = \varphi(\omega, p, q), \tag{8}$$

$$r(1-q^2) + t(1-p^2) = \psi(\omega, p, q), \tag{9}$$

where

$$\varphi = \frac{\varepsilon}{\sin\omega}(1-p^2)^{1/2}(1-q^2)^{1/2}; \tag{10}$$

$$\psi = \operatorname{ctg}\omega(1-p^2)(1-q^2)(3p^2+3q^2-4p^2q^2-2)+4\frac{\varepsilon pq}{\sin\omega}(1-p^2)^{3/2}(1-q^2)^{3/2}. \tag{11}$$

The integrability condition of the system (8), (9)

$$\frac{\partial^2 r}{\partial U\,\partial V} = \frac{\partial^2 s}{\partial U^2} \tag{12}$$

has the following form:

$$Ar^2 + Br + C = 0, \tag{13}$$

where

$$A = 10\varphi(1-p^2)^{-2}; \tag{14}$$

$$B = 20\varphi\operatorname{ctg}\omega\,\frac{1-2p^2}{1-p^2} - 40\frac{pq}{\sin^2\omega}; \tag{15}$$

$$C = 8pq(1-p^2)(4p^2+1)\frac{\cos\omega}{\sin^3\omega}+\frac{8\varphi}{\sin^2\omega}(4p^2+4q^2-2p^2q^2-1)\frac{1-p^2}{1-q^2}+2\varphi\ \mathrm{ctg}^2\,\omega(24p^4-28p^2+9). \tag{16}$$

From (9) and (13), r and t are determined as functions of ω, p, q, namely:

$$r = \frac{2}{\sin\omega}(1-p^2)^{3/2}(1-q^2)^{-1/2}(\varepsilon pq + p\sqrt{\Delta}) + \mathrm{ctg}\,\omega(1-p^2)(2p^2-1); \tag{17}$$

$$t = \frac{2}{\sin\omega}(1-q^2)^{3/2}(1-p^2)^{-1/2}(\varepsilon pq - p\sqrt{\Delta}) + \mathrm{ctg}\,\omega(1-q^2)(2q^2-1), \tag{18}$$

where

$$\Delta = (1-4p^2)(1-4q^2) - (\cos\omega\sqrt{1-p^2}\sqrt{1-q^2} + 3\varepsilon pq)^2; \tag{19}$$

$$\rho^2 = 1/5. \tag{20}$$

Finally, the integrability conditions

$$\frac{\partial r}{\partial V} = \frac{\partial s}{\partial U}, \qquad \frac{\partial t}{\partial U} = \frac{\partial s}{\partial V} \tag{21}$$

are reduced to the form

$$\begin{aligned} 4\varepsilon p\sqrt{\Delta}\,[q(1-p^2)+\varepsilon\theta] &= p\sigma(p,q,\theta), \\ -4\varepsilon p\sqrt{\Delta}\,[p(1-q^2)+\varepsilon\theta] &= q\sigma(p,q,\theta), \end{aligned} \tag{22}$$

where

$$\theta = \cos\omega\sqrt{1-p^2}\sqrt{1-q^2}, \tag{23}$$

$$\sigma = 2 - 5p^2 - 5q^2 + 8p^2q^2 - 6\varepsilon pq\,\theta - 2\theta^2. \tag{24}$$

It is easy to see that $\Delta \neq 0$. Indeed, in the case $\Delta = 0$, from equations (22) it follows that $\sigma = 0$, and from equations (19), (24) that $\cos^2\omega = 1$, which is impossible. Consequently, system (22) is equivalent to the system

$$p^2 + q^2 - 2p^2q^2 + \varepsilon\theta(p+q) = 0; \tag{25}$$

$$4\varepsilon p\sqrt{\Delta}(q-p)(1+pq) = (p+q)\sigma. \tag{26}$$

From this system p, q are determined as functions of ω:

$$p = p(\omega), \qquad q = q(\omega). \tag{27}$$

But

$$s = \frac{dp}{d\omega}\, q = \frac{dq}{d\omega}\, p, \tag{28}$$

therefore

$$p = Cq, \qquad C = \text{const}. \tag{29}$$

From equations (25), (26), (29) it follows that p and q are constant, which contradicts the assumption $K \neq 0$. Theorem 1 is proved.

By the well-known theorem of Bianchi ([1]), surfaces of zero curvature in elliptic space are a special case of displacement surfaces, i.e., surfaces admitting a representation $x = a(u)b(v)$, where a, b are quaternions.

In ([2]) it was proved that from the existence in elliptic space of displacement surfaces of constant curvature $K \neq 0$ there follows the existence in Euclidean space of translation surfaces of constant curvature K, and conversely. Hence Theorem 2 follows from Theorem 1.

In ([2]) it was proved that a Fossa surface of constant curvature $K \neq 0$ is at the same time a translation surface, and conversely; hence Theorem 3 follows from Theorem 1.

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CITED LITERATURE

1. L. Bianchi, *Lezioni di geometria differenziale*, 2, Bologna, 1924.

2. Ya. P. Blank, *Zap. Inst. matem. i mekh. Kharkovsk. gos. univ. i Kharkovsk. matem. obshch.*, **20**, 61 (1950).

Note: Figure translations are in progress. See original paper for figures.