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Abstract

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ON THE CONDITIONS FOR SOLVABILITY OF A BOUNDARY-VALUE PROBLEM FOR THE HEAT-CONDUCTION EQUATION

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1. Find a solution of the heat-conduction equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1}$$

in the domain D $(0 < x < \infty,\ -\infty < y < +\infty,\ 0 < t < T_0)$, satisfying the initial condition

$$u(x, y, 0) = 0 \tag{2}$$

and the boundary condition

$$\sum_{k=0}^{m} \sum_{j=0}^{k} a_{kj} \frac{\partial^k u}{\partial x^j\, \partial y^{k-j}} \bigg|_{x=0} = f(y, t), \tag{3}$$

where a_{kj} are constants, and $f(y, t)$ is a known function which, together with its derivative with respect to y, satisfies the inequality:

$$|f(y, t)|,\ |f'_y(y, t)| \le M e^{\delta^2 y^2}. \tag{4}$$

We shall seek a solution $u(x, y, t)$, continuous together with its derivatives up to order m with respect to x and y in $\overline{D}$, and such that

$$\left| \frac{\partial^k u}{\partial x^j\, \partial y^{k-j}} \right| \le M_1 e^{\delta^2 r^2} \quad (j = 0, 1, \dots, k;\ k = 0, 1, \dots, m), \tag{5}$$

where $r = \sqrt{x^2 + y^2}$; $M,\ M_1,\ \delta^2$ are certain constants, and T_0 is also a constant satisfying the inequality

$$T_0 < \frac{1}{4a^2\delta^2}. \tag{6}$$

In the present work we shall show that the posed problem does not always have a solution, in contrast to the one-dimensional problem ([1]), and we shall indicate conditions under which our problem has a solution.

2. It is obvious that if the posed problem has a solution, then it can always be expressed by the formula*

$$u(x,y,t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \psi(\eta,\tau) \frac{x}{4a^2\pi(t-\tau)^2} \exp\left[-\frac{x^2+(y-\eta)^2}{4a^2(t-\tau)}\right] d\eta, \tag{7}$$

where

$$\psi(\eta,\tau) = u(+0,\eta,\tau). \tag{8}$$

From the continuity of the derivatives of the function $u(x,y,t)$ up to order m with respect to x, y in $\overline{D}$ there follows the existence and continuity of the derivatives

* We assume that m is an even number. If m is an odd number, then instead of (7) we take the simple-layer potential.

$$\frac{\partial^{m-k/2}\psi}{\partial y^{m-k}\,\partial t^{k/2}} \quad (k = 0, 2, 4, \dots, m),$$

where

$$\left|F^j\left[\psi_\eta^{(k-j)}\right]\right| \le M_2 e^{\delta_2\eta^2} \quad \left(j = 0, 1, \dots, k; \quad k = 0, 1, \dots, \frac{m}{2}\right); \tag{9}$$

$$F^j\left[\psi_\eta^{(k-j)}\right]_{\tau=0} = 0 \quad \left(j = 0, 1, \dots, k; \quad k = 0, 1, \dots, \frac{m}{2}\right), \tag{10}$$

where

$$F = \frac{\partial}{\partial t} - a^2\frac{\partial^2}{\partial\eta^2}, \qquad F\left[F^{k-1}\right] = F^k, \qquad F^0 = 1, \tag{11}$$

and conversely, if the function $\psi(\eta,\tau)$ satisfies inequalities (9) and conditions (10), then, under condition (6), the function $u(x,y,t)$ defined by formula (7) satisfies equation (1), the initial condition (8), and conditions (5). Therefore, in (7) we choose the function $\psi(\eta,\tau)$ from the class of functions satisfying inequalities (9) and conditions (10) so that the function $u(x,y,t)$ satisfies the boundary condition (3).

First we consider the simplest problem, i.e., the problem with boundary condition

$$\sum_{k=0}^{m} a_{m,k} \left.\frac{\partial^m u}{\partial x^k \partial y^{m-k}}\right|_{x=0} = \varphi(y,t). \tag{12}$$

Differentiating (7) successively and taking (10) into account, we obtain

$$\left.\frac{\partial^{2n-1+k} u}{\partial x^{2n-1} \partial y^k}\right|_{x=0} = -\frac{1}{a^{2n}} \int_0^t d\tau \int_{-\infty}^{+\infty} F^n \left[\psi_\eta^{(k)}\right] g(y-\eta, t-\tau)\, d\eta; \tag{13}$$

$$\left.\frac{\partial^{2n+k} u}{\partial x^{2n} \partial y^k}\right|_{x=0} = -\frac{1}{a^{2n}} F^n \left[\psi_\eta^{(k)}\right], \tag{14}$$

where

$$g(y-\eta, t-\tau) = \frac{1}{2\pi a(t-\tau)} \exp\left[-\frac{(y-\eta)^2}{4a^2(t-\tau)}\right]. \tag{15}$$

Substituting the found limiting values (13) and (14) into (12), we obtain the following integro-differential equation:

$$\sum_{k=0}^{m/2} \frac{a_{m,2k}}{a^{2k}} F^k \left[\psi_y^{(m-2k)}\right] - \sum_{k=1}^{m/2} \frac{a_{m,2k-1}}{a^{2k}} \int_0^t d\tau \int_{-\infty}^{+\infty} F^k \left[\psi_\eta^{(m-2k+1)}\right] g(y-\eta, t-\tau)\, d\eta = \varphi(y,t). \tag{16}$$

3. To simplify the last equation we introduce the operators

$$F^{-1}[\psi] = \int_0^t d\tau \int_{-\infty}^{+\infty} \psi(\eta,\tau) G(y-\eta, t-\tau)\, d\eta, \qquad F^{-k} = F^{-1}\left[F^{-k+1}\right], \tag{17}$$

where

$$G(y-\eta, t-\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(y-\eta)^2}{4a^2(t-\tau)}\right].$$

By direct calculation we verify that the operators F^{-1} and F are mutually inverse under condition (10).

In passing, let us introduce one more operator:

$$F^{-1/2}[\psi] = \int_0^t d\tau \int_{-\infty}^{+\infty} \psi(\eta,\tau) g(y-\eta, t-\tau)\, d\eta. \tag{18}$$

It is easy to see that

$$F^{-1/2}\left[F^{-1/2}[\psi]\right] = F^{-1}[\psi]. \tag{19}$$

Applying these operators successively, we obtain the following formulas useful to us:

$$F^{-k}[\psi] = \int_0^t d\tau \int_{-\infty}^{+\infty} \psi(\eta,\tau) \frac{(t-\tau)^{k-1}}{(k-1)!} G(y-\eta, t-\tau)\, d\eta,$$

$$F^{-k-1/2}[\psi] = \int_0^t d\tau \int_{-\infty}^{+\infty} \psi(\eta,\tau) \frac{(t-\tau)^{k-1/2}}{\Gamma(k+1/2)} G(y-\eta, t-\tau)\, d\eta.$$

4. Apply the operator $F^{-m/2}$ to equation (16). Then we obtain the following equivalent equation

$$a_{m,m}\psi(y,t) + \sum_{k=1}^{m} (-1)^k a_{m,m-k} a^k \int_0^t d\tau \int_{-\infty}^{+\infty} \psi_\eta^{(k)}(\eta,\tau) \times$$

$$\times \frac{(t-\tau)^{k/2-1}}{\Gamma(k/2)} G(y-\eta, t-\tau)\, d\eta = a^m F^{-m/2}[\varphi] = \varphi_1(y,t). \tag{20}$$

This equation has been well studied by us ([2]).

In order that equation (20) have a solution in the class of ordinary functions satisfying (9) and (10), it is necessary and sufficient that the roots of the algebraic equation

$$\sum_{k=0}^{m} a_{m,k} z^k = 0 \tag{21}$$

lie between the branches of the hyperbola

$$y^2 - x^2 = 1 \tag{22}$$

in the complex z-plane.

Let $q_1, q_2, \dots, q_\nu$ be the roots of equation (21) with multiplicities $n_1, n_2, \dots, n_\nu$, and let

$$\frac{x^m}{\sum_{k=0}^{m} a_{m,k} x^k} = 1 + \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} \beta_{k,j} \frac{1}{(x-q_k)^j}. \tag{23}$$

If $q_1, q_2, \dots, q_\nu$ lie between the branches of the hyperbola (22), then the solution of equation (20) is written in the form

$$\psi(y,t)=\varphi_1(y,t)+\sum_{k=1}^{\nu}\sum_{j=1}^{n_k}\beta_{k,j}\int_0^t d\tau\int_{-\infty}^{+\infty} w_{q_k}^{(j-1)}(y-\eta,t-\tau;q_k)\,\varphi'_{1\eta}(\eta,\tau)\,d\eta, \tag{24}$$

where

$$w(y,t;q_k)=-\frac{1}{2\pi t}e^{-y^2/4a^2t}-$$

$$-\frac{aq_k}{2\sqrt{\pi t}\,(1+q_k^2)^{1/2}}\frac{\partial}{\partial y}\left[e^{-y^2/4a^2(1+q_k^2)t}\operatorname{erfc}\left(-\frac{q_k}{\sqrt{1+q_k^2}}\frac{y}{2a\sqrt{t}}\right)\right]. \tag{25}$$

5. Now consider the problem with boundary condition (3). To this end, we rewrite it in the following form:

$$\sum_{k=0}^{m}a_{m,k}\frac{\partial^m u}{\partial x^k\partial y^{m-k}}\bigg|_{x=0}=f(x,y)-\sum_{k=0}^{m-1}\sum_{j=0}^{k}a_{k,j}\frac{\partial^k u}{\partial x^j\partial y^{k-j}}\bigg|_{x=0}=\varphi_1(y,t). \tag{26}$$

On the basis of (13) and (14),

$$\begin{aligned}\varphi_1(y,t)&=a^m\int_0^t dt_1\int_{-\infty}^{+\infty}f(y_1,t_1)\frac{(t-t_1)^{m/2-1}}{\Gamma(m/2)}G(y-y_1,t-t_1)\,dy_1\\&-\sum_{k=0}^{m-1}\sum_{j=0}^{k}a_{k,j}a^{m-j}\int_0^t dt_1\int_{-\infty}^{+\infty}\psi_{y_1}^{(k-j)}(y_1,t_1)\frac{(t-t_1)^{m/2-j/2-1}}{\Gamma(m/2-j/2)}G(y-y_1,t-t_1)\,dy_1.\end{aligned} \tag{27}$$

Substituting (27) into (24), we again obtain the integro-differential equation

$$\psi(y,t)=f_1(y,t)-\sum_{n=0}^{m-1}\sum_{i=0}^{n}a_{n,i}\int_0^t d\tau\int_{-\infty}^{+\infty}\psi_\eta^{(n-i)}(\eta,\tau)K_i(y-\eta,t-\tau)\,d\eta, \tag{28}$$

where

$$f_1(y,t)=\int_0^t d\tau\int_{-\infty}^{+\infty}f(\eta,\tau)K_0(y-\eta,t-\tau)\,d\eta,$$

$$K_i(y-\eta,t-\tau) = \frac{a^{m-i}(t-\tau)^{(m-i)/2-1}}{\Gamma((m-i)/2)} G(y-\eta,t-\tau) - a^{m-i}\sum_{k=1}^{\nu}\sum_{j=1}^{n_k}\beta_{kj}\frac{\partial^j}{\partial\eta\,\partial q_k^{j-1}}\int_0^t dt_1\int_{-\infty}^{+\infty}\frac{(t_1-\tau)^{(m-i)/2-1}}{\Gamma((m-i)/2)} \times \omega(y-y_1,t-t_1;q_k)G(y_1-\eta,t_1-\tau)\,dy_1. \quad (29)$$

Transferring the derivatives from the unknown function in (28) onto K_i, we obtain the final integral equation

$$\psi(y,t) = f_1(y,t) + \int_0^t d\tau\int_{-\infty}^{+\infty} K(y-\eta,t-\tau)\psi(\eta,\tau)\,d\eta, \quad (30)$$

where

$$K(y-\eta,t-\tau) = \sum_{n=1}^{m-1}\sum_{i=0}^{n}(-1)^{n+i}a_{n,i}\frac{\partial^{n-i}}{\partial\eta^{n-i}}K_i(y-\eta,t-\tau).$$

A detailed investigation shows that

$$|K(y-\eta,t-\tau)| \le \frac{M}{t-\tau}e^{-\delta_1^2(y-\eta)^2/(t-\tau)}. \quad (31)$$

Consequently, equation (30) can be integrated by the method of successive approximations.

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Note: Figure translations are in progress. See original paper for figures.