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Abstract

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HYDROMECHANICS

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ON THE ENERGY SPECTRUM OF A TURBULENT FLOW OF AN INCOMPRESSIBLE FLUID

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1. The principal result of the modern theory of a locally homogeneous and locally isotropic turbulent flow of an incompressible fluid is the well-known Kolmogorov-Obukhov "2/3 law" (1) and its analogue in spectral language—the "5/3 law" (2):

$$E(p) = C\varepsilon^{2/3}p^{-5/3}. \tag{1}$$

Here $E(p)$ and ε are, respectively, the one-dimensional spectral density of kinetic energy and the dissipation of kinetic energy, calculated per unit mass of the fluid; p is the wave number; C is a constant of order unity. Formula (1) is valid in the inertial interval of wave numbers:

$$L^{-1} \ll p \ll \lambda_0^{-1} = \varepsilon^{1/4}\nu^{-3/4},$$

where L is the characteristic external scale of turbulence; λ_0 is the inner scale of turbulence (1); ν is the kinematic viscosity. For a number of problems it is of interest to determine the energy spectrum at smaller scales ($p\lambda_0 > 1$). In particular, this is important in the study of turbulent fluctuations of electron density in the ionosphere, where λ_0 reaches tens and hundreds of meters.

Townsend (3) made an attempt to determine the form of the spectrum in the region of large wave numbers. He assumed that small-scale fluctuations of vorticity are concentrated in spatially separated stationary layers or lines of small thickness. A criticism of such a representation as applied to temperature fluctuations in a turbulent flow was given in Batchelor' s work (4). We shall proceed in a somewhat different way, using (with refinements) the method developed in (4) for studying the temperature field. We shall try to trace how small-scale fluctuations of vorticity, coming from motions of larger scales, decay under the action of viscosity, while at the same time experiencing the influence of the deformation of a fluid element of size $\sim \lambda_0$. The form of the spectrum will be determined as invariant with respect to such an evolution. We shall not require spatial separation of individual fluctuations, which in the process of change may in principle be superposed on one another. Just as was in fact implied in (^{3}), we shall neglect the nonlinear mechanism of energy transfer. This is justified by the sharp character of the decrease of the spectral density of kinetic energy (8) in the interval of wave numbers under consideration ($p\lambda_0 \gg 1$), owing to which Fourier components corresponding to substantially different wave numbers are incommensurable in amplitude.

2. Let us consider the equation of transport and diffusion of vorticity in an incompressible fluid:

$$\frac{\partial \Omega_i}{\partial t} + v_k \frac{\partial \Omega_i}{\partial x_k} - \Omega_k \frac{\partial v_i}{\partial x_k} = \nu \Delta \Omega_i, \qquad \Omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j}. \tag{2}$$

Here v_i and Ω_i are, respectively, the components of velocity and vorticity; summation over twice-repeated indices is understood. This equation is obtained by applying the curl operation to the Navier–Stokes equation. In a frame of reference moving and rotating together with a fixed fluid element, at distances smaller than λ_0, the motion reduces to pure deformation, since at such distances the velocity may be regarded as a linear function of the coordinates: $v_i = \alpha_{ik} x_k$. The tensor α_{ik} is symmetric (the antisymmetric part is eliminated by passing to a rotating frame of reference) and can be reduced to principal axes, to which there will correspond the principal values of the deformation rates α_i. From the condition of incompressibility we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{3}$$

We see that the α_i must have different signs. Let α_1 be the greatest positive deformation rate (stretching), and α_3 the greatest negative deformation rate (compression). In (^{5}), from observations of the evolution of thermal spots in a turbulent flow, the conclusion was drawn that the orientation of the principal axes of deformation and the principal values of the deformation rates change little over times of the order $\tau_0 = \varepsilon^{-1/2} \nu^{1/2}$ (the reciprocal of the deformation rates (^{9})). We shall suppose for the time being that over times $\sim \tau_0$ the principal axes of deformation do not have time to rotate appreciably relative to the fluid element (and do not change places). Such an assumption is also consistent with the picture of the behavior of a dyed volume of fluid in a turbulent flow (for the case of a two-dimensional flow, see (^{6})).

Choosing the principal axes of deformation as the coordinate axes, one can obtain from equation (2), with the initial condition $\Omega_{0i} = A_{0i} \sin(p_0 x)$:

$$\Omega_i = A_i \sin(px), \qquad A_i = A_{0i} \exp\left\{ \int_0^t (\alpha_i - \nu p^2)\, dt \right\}, \qquad p_i = p_{0i} \exp\left\{ -\int_0^t \alpha_i \, dt \right\}. \tag{4}$$

From these formulas it is evident that over times $\sim \tau_0$ the vorticity has time to adjust to the deformation of the fluid element in such a way that $A^2 \approx A_1^2$, $p^2 \approx p_3^2$*, i.e., the vorticity is oriented along the axis of maximum stretching and becomes dependent only on the coordinate along the axis of maximum compression**. Let us introduce the quantity $E(p,t) = A^2(p,t)/p^2\delta p$, which, after probabilistic averaging, will be proportional to the spectral density of kinetic energy (δp is the interval of wave numbers to which the amplitude A corresponds; the factor p^2 arises in passing from the spectral density of vorticity to the spectral density of kinetic energy). Considering the vorticity already "adjusted," from (4) we have:

$$dA^2 = 2A^2(\alpha_1 - \nu p^2)\,dt, \qquad dp = -\alpha_3 p\,dt, \qquad d\delta p = -\alpha_3 \delta p\,dt,$$

whence

$$p\ln E = (-2\alpha_1\alpha_3^{-1} - 3 + 2\nu\alpha_3^{-1}p^2)\,dp. \tag{5}$$

After probabilistic averaging, putting

$$k = -\overline{a_1 a_3^{-1}}, \qquad a = -\tau_0^{-1}\overline{a_3^{-1}}, \tag{6}$$

we obtain:

$$\widetilde{E}(p) = \exp\{\overline{\ln E(p)}\} \sim p^{2k-3}\exp\{-at_0\nu p^2\}, \tag{7}$$

where the wavy bar (which we shall omit in what follows) denotes geometric averaging*. The proportionality coefficient in (7) is determined from dimensional considerations (a combination of ε and ν), after which we have

$$E(p) = C_1\varepsilon^{2/3}p^{-5/3}(p\lambda_0)^{2k-4/3}\exp\{-a(p\lambda_0)^2\} \qquad (p\lambda_0 \gg 1), \tag{8}$$

where C_1 is a constant of order unity.

3. Let us give some estimates of the parameters of the problem. From the inequalities $0 \le a_1 \ge a_2 \ge a_3 \le 0$ and condition (3) we have $0.5 \le -a_1 a_3^{-1} \le 2$; consequently, the parameter k also lies within the same limits. Further, in a locally homogeneous and locally isotropic turbulent flow the following relations hold:

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* We do not consider here the statistically unlikely cases $A_{01} = 0$ or $p_{03} = 0$.

** For $\alpha_1 = \alpha_2$, for an "adjusted" vortex $A^2 \approx A_1^2 + A_2^2$. The subsequent formulas remain valid for this case.

$$\overline{a_1^2} + \overline{a_2^2} + \overline{a_3^2} = \frac{1}{2}\tau_0^{-2}; \tag{9}$$

$$S_0 = \overline{\left(\frac{\partial v_1}{\partial x_1}\right)^3} \left[\overline{\left(\frac{\partial v_1}{\partial x_1}\right)^2}\right]^{-3/2} = \frac{24\sqrt{15}}{7}\tau_0^3 \overline{a_1 a_2 a_3}. \tag{10}$$

Here S_0 is the skewness coefficient at scales $\sim \lambda_0$; $\partial v_1/\partial x_1$ is the longitudinal derivative of the velocity in any direction fixed in space. From (9) and (3) it follows that $a \geq \sqrt{3}$. Known physical considerations concerning the transfer of energy from larger-scale velocity pulsations to smaller-scale ones and experimental data ([7],[8]) indicate that the skewness coefficient is negative. If, in a first approximation, we neglect the correlation between the bounded quantity $-a_1 a_3^{-1}$ and the random quantity a_3, then from the condition $S_0 < 0$ it will follow that $k < 1$. Finally, if, taking into account the signs of the quantities a_1 and a_3, one interchanges the algebraic operations and the averaging operations, then from (9), (10), and (3) we obtain the following approximate formulas:

$$a \simeq 2\,(k^2 - k + 1)^{1/2}; \tag{11}$$

$$S_0 \simeq -\frac{3\sqrt{15}}{7}\,k(1-k)(k^2 - k + 1)^{-3/2}. \tag{12}$$

Let us note that, according to (12), $|S_0| \leq 2\sqrt{5}/7 \simeq 0.64$ (the maximum value is attained at $k = 0.5$).

As an additional hypothesis one may require that, in the inertial interval of wave numbers, formula (8) give the same dependence as (1). In this case we obtain the value $k = 2/3$, which lies in the indicated

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* Strictly speaking, in deriving (7) we use the inequality

$$\left| a_3^{-1} \frac{\partial}{\partial t} \ln E \right| \ll 1.$$

This inequality must be proved proceeding from the fact that the period of variation of the quantity a_3, which characterizes the motion of scale $\sim \lambda_0$, is of order τ_0 and, in the interval of wave numbers under consideration, is much greater than $(\nu p^2)^{-1}$, the characteristic period of variation of the quantity $E(p, t)$. (In connection with this, there arises an interesting mathematical problem of estimating the mutual correlation of two stationary random functions taken at one point, in the case of incommensurability of their autocorrelation scales.) If, following ([3],[4]), a_3 is taken to be constant, then the left-hand side of the inequality vanishes identically by virtue of the stationarity condition, since differentiation with respect to time may be taken outside the averaging sign.

above. In this case formula (8) may be used to describe the spectrum over the entire equilibrium interval ($pL \gg 1$), and the constant $C_1 = C$ is determined from the normalization condition

$$2\nu \int_0^\infty p^2 E(p)\, dp = \varepsilon, \qquad C = \frac{a^{2/3}}{\Gamma(2/3)}, \tag{13}$$

where $\Gamma(x)$ is the gamma function. For $k = 2/3$, from (11)–(13) we have $a = 2\sqrt{7}/3 \simeq 1.76$, $S_0 \simeq -0.54$, $C \simeq 1.08$. The constant C can be related to the asymmetry coefficient measured in the inertial interval (where it is constant). On the basis of experimental data [7,8] we obtain $C = 1.04 \div 1.34$, which agrees with the value found above. Thus, the following working formula may be proposed for the energy spectrum over the entire equilibrium interval:

$$E(p) = \frac{a^{2/3}}{\Gamma(2/3)}\, \varepsilon^{2/3} p^{-5/3} \exp\{-a(p\lambda_0)^2\}, \qquad a = \frac{2\sqrt{7}}{3} \quad (pL \gg 1). \tag{14}$$

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Note: Figure translations are in progress. See original paper for figures.