ON THE MOTION OF A BODY OF VARIABLE MASS WITH CONSTANT EXPENDITURE OF POWER IN A GRAVITATIONAL FIELD

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Abstract

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ON THE MOTION OF A BODY OF VARIABLE MASS WITH CONSTANT EXPENDITURE OF POWER IN A GRAVITATIONAL FIELD

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In the present work the general case is investigated of optimization of the reactive motion of a body of variable mass in the gravitational field of two centers, with a constant expenditure of power $N = \text{const}$, and the corresponding variational problem is considered.

1. For a prescribed trajectory of motion, the acceleration due to reactive thrust is a given quantity

$$a = a(t) = -\frac{V}{m}\frac{dm}{dt},$$

where V is the exhaust velocity of the working body, and the useful reactive power is written in the form

$$N = -\frac{dm}{dt}\frac{V^2}{2};$$

hence we obtain

$$\frac{a^2}{2N} = -\frac{1}{m^2}\frac{dm}{dt}. \tag{1}$$

Integrating equation (1), we arrive at the following law for the change of the body' s weight with time t:

$$G = G_0 \bigg/ \left(1 + \frac{G_0}{2Ng}\int_0^T a^2\,dt\right), \tag{2}$$

where g is the terrestrial acceleration of gravity, G_0 is the initial weight.

Let us denote the specific weight of the power source by $\alpha = G_N/N$. The relative total initial weight of the reserve of expelled mass G_M and of the power source G_N will be equal to

$$\overline{G} = \frac{G_M + G_N}{G_0} = \alpha \frac{N}{G_0} + 1 - 1 \bigg/ \left(1 + \frac{G_0}{2Ng} \int_0^T a^2\, dt \right). \tag{3}$$

For a prescribed function $a(t)$, the quantity $\overline{G}$ has a minimum

$$\overline{G}_{\min} = 2\sqrt{\Phi} - \Phi \quad \text{when} \quad (G_N/G_0)_{\text{opt}} = (\alpha N/G_0)_{\text{opt}} = \sqrt{\Phi} - \Phi,$$

where

$$\Phi = \frac{\alpha}{2g} \int_0^T a^2\, dt.$$

In the case of a multistage reduction of power with a corresponding reduction of weight, the maximum relative useful weight is expressed by the formula

$$\overline{G}_{\text{p. max}} = (1 + \Phi_1 - 2\sqrt{\Phi_1}) \prod_{i=2}^{n} \left(\frac{1 - \Phi_i}{1 + \Phi_i} \right)^2, \tag{4}$$

where

$$\sum_{i=1}^{n} \Phi_i = \Phi$$

is prescribed. The optimal relation between the Φ_i can be obtained by differentiating (4).

The variation of $\overline{G}_{\min}$ and G_N/G_0 as functions of Φ is given in Fig. 1.

2. It follows from Sec. 1 that the minimum of $\overline{G}$ requires a minimum of the integral.

$$\int_0^T a^2\, dt.$$

As an application, let us consider the simplest plane motion along a gently sloping spiral in a central gravitational field for small values of a.

We write the equation of motion in polar coordinates r, ψ in the form

$$a_r = \dot{v}_r - \frac{v_\psi^2}{r} + \frac{g_0 R_0^2}{r^2},$$

$$a_\psi = \frac{1}{r}\frac{d}{dt}(r v_\psi), \qquad v_r = \dot{r}, \qquad v_\psi = r\dot{\psi}, \tag{5}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

where g_0 is the acceleration of gravity at the radius R_0.

For realizing spiral motion, put $a_\psi = k(t)g_0,\ a_r = v_r^*$; then

$$v_\psi = R_0\sqrt{g_0/r}, \qquad k(t) = -R_0^2 \frac{d}{dt}\left(\frac{1}{rv_\psi}\right).$$

Fig. 1

After integration, assuming that at $t = 0$, $r = R_0,\ v_\psi = v_0 = \sqrt{g_0 R_0}$, we obtain the following law of spiral motion:

$$\frac{r}{R_0} = \left[1 - \frac{\int_0^t k(t)\,dt}{R_0/v_0}\right]^2. \tag{6}$$

Let us note that in this motion the transition from a given radius R_0^* to a given radius R_1 in a given time is determined by the integral

$$\int_0^T k(t)\,dt.$$

To ensure, in this case, the minimum of $\bar{G}$, a minimum of the integral

$$\int_0^T k^2(t)\,dt$$

is required. According to the Cauchy–Bunyakovsky inequality,

$$T\int_0^T k^2(t)\,dt \geq \left[\int_0^T k(t)\,dt\right]^2,$$

therefore, in the spiral motion under consideration, the minimum $\bar{G}$ is attained when $k(t) = \text{const}$.

Fig. 2

3. Let us consider the problem of the optimal transfer of a body of variable mass in time T between two given points (with given velocities) with minimum $\bar{G}$. This leads to the variational problem of finding the extremals of the integral
$$I = \int_0^T a^2(t)\,dt$$
in the class of continuous functions $a(t)$.

Let us consider plane motion in the gravitational field of two gravitating centers, one of which is at rest, while the other rotates about it with angular velocity ω along a circle of radius r_0. To study the character of the motion in the region of predominant influence of one of the centers, it is expedient to choose a coordinate system associated with this center (see Fig. 2). The equations of motion in the chosen coordinate systems can

* It can be shown that in this case a_r is of order ka_ψ, i.e., for small k and for total energy of motion not close to zero, the radial component is negligible.

write in a unified form:

$$\ddot{r}_i - r_i(\dot{\psi}_i + \omega)^2 = a_{r_i} - \frac{k_i}{r_i^2} - \mathfrak{R}_i; \qquad (7)$$

$$r_i\ddot{\psi}_i + 2\dot{r}_i(\dot{\psi}_i + \omega) = a_{\psi_i} - \Psi_i, \qquad i = 1, 2, \qquad (8)$$

where

$$\mathfrak{R}_1 = \frac{k_2}{r_0^2}\cos\psi_1 - \frac{k_2}{r_2^2}\cos(\psi_1 - \psi_2), \quad \mathfrak{R}_2 = \frac{k_1}{r_1^2}\cos(\psi_1 - \psi_2),$$

$$\Psi_1 = \frac{k_2}{r_0^2}\sin\psi_1 + \frac{k_2}{r_2^2}\sin(\psi_1 - \psi_2), \quad \Psi_2 = \frac{k_1}{r_1^2}\sin(\psi_1 - \psi_2),$$

k_1 and k_2 are the gravitational constants.

It is required to find trajectories $r(t)$ and $\psi(t)$ that minimize the integral

$$I = \int_0^T \left\{ \left[\ddot{r}_i - r_i(\dot{\psi}_i + \omega)^2 + \frac{k_i}{r_i^2} - \mathfrak{R}_i \right]^2 + \left[r_i\ddot{\psi}_i + 2\dot{r}_i(\dot{\psi}_i + \omega)\Psi_i \right]^2 \right\} dt. \quad (9)$$

We shall represent the Euler equations of this variational problem in the form

$$\dot{a}_{r_i} = \frac{1}{v_{r_i}} \left[\frac{a_{r_i}^2 + a_{\psi_i}^2}{2} + a_{r_i} \left(\frac{v_{\psi_i}^2}{r_i} - \frac{k_i}{r_i^2} \right) - \lambda_i - v_i \frac{v_{\psi_i}}{r_i} \right], \tag{10}$$

$$\dot{a}_{\psi_i} = \frac{1}{r_i} \left(a_{\psi_i} v_{r_i} - 2 a_{r_i} v_{\psi_i} + v_i \right); \tag{11}$$

$$\dot{v}_i = a_{\psi_i} \Psi'_{\psi_i} + a_{r_i} \mathfrak{R}'_{\psi_i}; \tag{12}$$

$$\dot{\lambda}_i = -v_{r_i} \left(a_{\psi_i} \Psi'_{r_i} + a_{r_i} \mathfrak{R}'_{r_i} + \frac{\Psi_i}{r_i} a_{\psi_i} \right) - v_i \frac{\Psi_i}{r_i} - \dot{a}_{r_i} \mathfrak{R}_i - \frac{v_{\psi_i}}{r_i} \left(v_i - 2 a_{r_i} \Psi_i \right);$$

$$v_{r_i} = \dot{r}_i, \qquad v_{\psi_i} = r_i (\dot{\psi}_i + \omega). \tag{14}$$

The origin of the functions v and λ will be clarified by considering motion in a central field ($k_2 = 0;\ \omega = 0$). In this case $\mathfrak{R}_1 = 0$ and $\Psi_1 = 0$, i.e. $\dot{v} = 0$ and $\dot{\lambda} = 0$; $v \equiv$ const and $\lambda \equiv$ const are integrals of the system. Let us simplify the equations of motion (7), replacing differentiation with respect to time by differentiation with respect to radius:

$$v'_r v_r - \left(v_\psi^2 / r - k / r^2 \right) = a_r, \qquad v'_\psi v_r + v_r v_\psi / r = a_\psi,$$

and consider the following variational problem: to find the trajectory providing

$$\min \int_{r_1}^{r_2} a^2 \, \frac{dr}{v_r}$$

under the additional isoperimetric conditions: a prescribed time of displacement from r_1 to r_2

$$\left(T = \int_{r_1}^{r_2} \frac{dr}{v_r} \right)$$

and a prescribed polar angle of displacement

$$\Delta\psi = \int_{r_1}^{r_2} \frac{v_\psi}{r} \frac{dr}{v_r},$$

i.e., we find the extremals of the integral

$$\int_{r_1}^{r_2} \left(\frac{a_r^2 + a_\psi^2}{v_r} + \varkappa_1 \frac{1}{v_r} + \varkappa_2 \frac{v_\psi}{r v_r} \right) dr. \tag{15}$$

After carrying out the differentiation and returning to the variable t ($dt = dr/v_r$), we obtain expressions for $\dot{a}_r$ and $\dot{a}_\psi$ coinciding with (10) and (11) for $\varkappa = 2\lambda$ and $\varkappa_2 = 2v^*$.

* In work (1), devoted to the optimal programming of the thrust vector during motion in a central field with $N =$ const, the constant v is omitted, which narrows the class of possible solutions.

Thus, in a central field the functions λ and ν have the meaning of constants under the conditions of isoperimetry.

4. Thus, the solution of the problem posed in Sec. 3 reduces to the joint integration of two systems of equations (7) and (10)–(14).

As an example, let us consider motion in a force-free field: $k_1 = 0$, $k_2 = 0$. In this case equations (7), (10)–(14) in a rectangular coordinate system will be written in the form

$$\frac{d^2\bar{r}}{dt^2} = \bar{a}; \tag{16}$$

$$\frac{d^4\bar{r}}{dt^4} = 0, \tag{17}$$

whence

$$\bar{a} = \bar{b}_1 t + \bar{b}_2, \qquad \bar{r} = \frac{1}{6}\bar{b}_1 t^3 + \frac{1}{2}\bar{b}_2 t^2 + \bar{b}_3 t + \bar{b}_4. \tag{18}$$

The vectors $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3$, and $\bar{b}_4$ are determined from the boundary conditions according to the prescribed coordinates and velocities at two points and the time of transfer T between them. Let us note that in this case, along the optimal trajectory, the acceleration $\bar{a}$ varies linearly with time (see (18)), while in planar motion the ratio of the acceleration components is a fractional-linear function of time, as in the work[2], where the case of constant reactive thrust is investigated.

5. Let us investigate the singularities of the system of equations obtained. For simplicity we consider motion in a central field. As is seen from equation (10), when the radial velocity vanishes, $v_r = 0$, $\dot{a}_r \to \infty$, unless the numerator of the right-hand side of this equation also vanishes at these points. Eliminating a_r from equations (7), we arrive at an equation linear with respect to $\frac{1}{2}\dot{v}_r^2$; its solution is written in the form

$$\frac{1}{2}\dot{v}_r^2 = cv_r + v_r \int \frac{\dot{f}v_r - \frac{1}{2}f^2 + h}{v_r^2}\, dv_r, \tag{19}$$

where $f = v_\psi^2/r - k/r^2$, $h = \frac{1}{2}a_\psi^2 - \lambda - \nu v_\psi/r$, and c is the constant of integration.

The integrand in (19) can be expanded at the point $v_r = 0$ in powers of v_r; in doing so the derivatives d^k/dv_r^k will be finite in the case of finiteness of $\dot{v}_r$. Integrating this expansion and letting v_r tend to zero, we find, in the case of finite c,

$$\left(\frac{1}{2}\dot{v}_r^2\right)_{v_r=0} = \left(\frac{1}{2}f^2 - h\right)_{v_r=0},$$

or, replacing f and h by their values and substituting $\dot{v}_r = a_r + f$, we obtain

$$\frac{1}{2}a_r^2 + a_r f + \frac{1}{2}f^2 = \frac{1}{2}f^2 - \frac{1}{2}a_\psi^2 + \lambda + \nu\dot{\psi}. \tag{20}$$

It is evident that the condition of finiteness of $\frac{1}{2}\dot{v}_r^2$, and hence also of a_r, consists in the vanishing of the numerator of the expression $\dot{a}_r$ at the point $v_r = 0$, and the point itself is a singular point of the "node" type.

As a result of differentiating $\frac{1}{2}\dot{v}_r^2$ with respect to v_r, we obtain

$$\ddot{v}_r = c + \dot{f} + \int \frac{1}{v_r}\left[\dot{f} - \frac{d}{dv_r}\left(\frac{1}{2}f^2 - h\right)\right] dv_r. \tag{21}$$

As $v_r \to 0$, the last integral tends to zero (this can be shown by expanding the integral in a neighborhood of the point $v_r = 0$ and letting $v_r \to 0$). Hence the value of the constant c is determined—this is the value of a_r at $v_r = 0$.

The investigation carried out makes it possible to use methods of numerical integration for solving the problem in the general case.

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