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Abstract

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MATHEMATICS

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ON THE QUESTION OF AN ISOMETRIC IMMERSION OF A TWO-DIMENSIONAL RIEMANNIAN MANIFOLD HOMEOMORPHIC TO A SPHERE INTO A THREE-DIMENSIONAL RIEMANNIAN SPACE

The question of an isometric immersion in the large of a two-dimensional Riemannian manifold homeomorphic to a sphere into a three-dimensional Riemannian space was considered by the author in paper ([1]). In that paper a solution of the question was obtained under the assumption that the Gaussian curvature of the immersed manifold is greater than a certain constant depending on the curvature of the space.

In the author' s note ([2]), this result was improved and, for spaces of nonpositive curvature, assumed in a certain sense a definitive form.

At present the author has succeeded in obtaining a complete solution of the question in the form of the following theorem.

Theorem 1. Let R be a complete three-dimensional Riemannian space, and let M be a closed Riemannian manifold homeomorphic to a sphere, with Gaussian curvature everywhere greater than a certain constant c (greater than, less than, or equal to zero).

Then, if the curvature of the space R is everywhere less than c, M admits an isometric immersion into R in the form of a regular surface F.

Moreover, this immersion can be carried out so that a given two-dimensional element α of the manifold M (a point and a pencil of directions in it) coincides with a prescribed two-dimensional element α', isometric to α, in the space R, and the surface F is situated on the prescribed side of the plane element α'.

If the metric of the space R and of the manifold M are k times differentiable ($k \geqslant 6$), then the surface F is differentiable at least $k-1$ times. If the metric of the space R and of the manifold M are analytic, then the surface F is analytic.

The incompleteness of the results obtained in the author' s previous papers ([1],[2]) is explained mainly by the difficulties arising in obtaining a priori estimates for the normal curvatures of a regular surface in a Riemannian space, depending only on the metric of the surface and the metric of the space. In paper ([1]) such estimates are obtained under the assumption that the Gaussian curvature of the surface is greater than a certain constant depending on the curvature of the space, while in paper ([2]) under the assumption that the Gaussian curvature of the surface is simply greater than the curvature of the space, but the curvature of the space must be everywhere nonpositive. Without going into certain details, for a complete solution of the immersion problem in the form in which we have formulated it, it is sufficient to prove the following proposition concerning a priori estimates.

Let F be a locally strictly convex surface in a Riemannian space R with regular metric (regularity of the surface itself is not assumed). Let F_n be a sequence of regular surfaces converging to F, and suppose that the metrics of these surfaces converge to the metric of F in the class C^k ($k \geqslant 6$).

Then, if the Gaussian curvature F is strictly greater than the curvature of the space, then for the normal curvatures F_n, provided F_n is sufficiently close to F, there is an upper estimate independent of n.

We prove this assertion by contradiction as follows. If the assertion is false, then, without loss of generality, one may assume that on the surface F_n there is a point P_n at which the normal curvature in one of the directions is greater than n. Likewise, without loss of generality, one may assume that the points P_n converge to a certain point P_0 of the surface F.

Take a ball G_0 with center P_0 so small that any two of its points are joined by a unique shortest path in R. Let ω_n be the part of the surface F_n contained in the ball G_0, and let ω_0 be the part of F inside this ball. Join an arbitrary point A of the surface ω_0 with the point P_0. Let $\vartheta(A)$ be the angle which the inner normal to ω_0 at the point A forms with the direction of the shortest path AP_0. By the strict convexity of ω_0, the angle $\vartheta(A) < \pi/2$, and for all points A that are at distance at least $\varepsilon > 0$ from P_0, $\vartheta(A) < \pi/2 - \eta(\varepsilon)$, where $\eta(\varepsilon)$ is positive.

The point P_0 on the surface ω_0 cannot be conical. Therefore there passes through it a certain geodesic γ_0 tangent to the surface ω_0. Draw a semigeodesic γ' from the point P_0 into the body whose boundary is the surface ω_0, so that at P_0 it forms with the geodesic γ_0 an angle less than $\eta(\varepsilon)/2$. On the semigeodesic γ' take a point P'. If the point P' is sufficiently close to P_0, and in the definition of the angle $\vartheta(A)$ one takes the point P' instead of P_0, then this angle, for points A removed by a distance greater than ε from P_0, will also be less than $\pi/2 - \eta(\varepsilon)$. By virtue of the convergence of ω_n to ω_0, for sufficiently large n the angle $\vartheta(A)$, determined for the surface ω_n and the point P', will likewise be less than $\pi/2 - \eta(\varepsilon)$ when $AP_0 < \varepsilon$, while for $A \equiv P_n$ this angle is greater than $\pi/2 - \eta(\varepsilon)/2$.

We now define on the surface ω_n the function

$$\overline{w}(X) = \frac{\overline{\varkappa}(X)}{(\cos\vartheta(X))^{\mu}},$$

where $\overline{\varkappa}(X)$ is the maximal normal curvature at the point X; $\vartheta(X)$ is the angle formed by the inner normal to the surface ω_n with the direction of the semigeodesic XP' at the point X, and μ is a certain positive constant. If ω_n is sufficiently close to ω_0, then for every point X satisfying the condition $XP_0 > \varepsilon$,

$$\overline{w}(X) < \overline{w}(P_n).$$

Indeed, by the definition of the point P_n, $\overline{\varkappa}(X) \leqslant \overline{\varkappa}(P_n)$. Further, $\vartheta(X) < \pi/2 - \eta(\varepsilon)$, while $\vartheta(P_n) > \pi/2 - \eta(\varepsilon)/2$. Hence $\vartheta(X) < \vartheta(P_n)$, and, consequently, $\overline{w}(X) < \overline{w}(P_n)$. Since for $XP_0 > \varepsilon$ we have $\overline{w}(X) < \overline{w}(P_n)$, the maximum of $\overline{w}(X)$ is attained at a certain point X_0 whose distance from the point P_0 is not greater than ε, and therefore whose distance from the point P' is not greater than $\varepsilon + P_0P'$.

Now we introduce, in a neighborhood of the point P', a polar geodesic coordinate system with pole at P'. In these coordinates the function $\overline{w}$ admits a simple expression. The maximum of $\overline{w}$ at the point X_0 is studied with the aid of the equations of isometric immersion (the analogue of the Darboux equations (1)). This study, accompanied by rather complicated calculations, makes it possible to establish estimates independent of n for the function $\overline{w}$ at the point X_0. At the same time an estimate is obtained for $\overline{\varkappa}(P_n)$, and thus we arrive at a contradiction, since by assumption $\overline{\varkappa}(P_n) > n$.

Theorem 1 on the possibility of an isometric immersion is supplemented by a theorem on unique determination and isometric transformations.

Theorem 2. The surface F, whose existence is asserted by Theorem 1, is determined uniquely by the two-dimensional element a'.

Theorem 3. If in a complete Riemannian space R with curvature in two-dimensional areas less than c, two regular isometric surfaces F_1 and F_2, homeomorphic to the sphere, are given with Gaussian curvature greater than c, then one surface admits a continuous bending into the other.

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REFERENCES

1. A. V. Pogorelov, *Some Questions of Geometry in the Large in Riemannian Space*, Kharkov University Press, 1957.

2. A. V. Pogorelov, DAN, **137**, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.