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Abstract

Full Text

MATHEMATICS

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ON THE THEORY OF MULTIPLICATIVE GROUPS OF DIVISION RINGS

(Presented by Academician P. S. Aleksandrov, 22 X 1960)

§ 1. Let K be an associative noncommutative division ring, and let Z be the center of the division ring K . The set of all nonzero elements of the division ring K forms a group under multiplication, which we shall denote by K^* . If Z^* is the set of all nonzero elements of Z , then Z^* is the center of the group K^* . In ⁽¹⁾ the author proved that every locally nilpotent normal divisor of the group K^* is contained in the center Z^* , and, consequently, the factor group K^*/Z^* has no nontrivial locally nilpotent normal divisors. In particular, K^* is not an RN^* -solvable group, i.e. K^* does not possess an ascending solvable normal series. Indeed, every RN^* -solvable normal divisor of the group K^* is contained in the center Z^* .

In the present note it is proved that these results are in a certain sense final. The following general theorem holds:

Theorem 1. *Every division ring K can be embedded in a division ring D , whose multiplicative group D^* possesses an infinite descending invariant series*

$$D^* \supset A_1 \supset A_2 \supset A_3 \supset \dots, \quad \bigcap_{j=1}^{\infty} A_j = C^*,$$

where: 1) C is the center of the division ring D ; 2) $[A_1, A_j] \subseteq A_{j+1}$, $j = 1, 2, \dots$, where $[A_1, A_j]$ is the mutual commutant of the subgroups A_1 and A_j ; 3) D^*/A_1 is isomorphic to K^*/Z^* .

Proof. Denote by D the ring of formal power series of the form $\sum_{i \geq m} a_i t^i$, where $a_i \in K$, m is an integer, and the indeterminate t commutes with all elements of the division ring K . Then D is a division ring, as a special case of the Hilbert division rings defined in ⁽²⁾ (pp. 187-188). The center C of the division ring D consists of all series of the form $\sum_{i \geq m} z_i t^i$, where $z_i \in Z$. It is clear that D contains K .

We now show that the division ring D has all the properties formulated in the theorem. If $a = \sum_{i \geq m} a_i t^i$ and $a_m \neq 0$, then we shall agree to call a_m the first coefficient of the element a , and the coefficients a_{m+1}, a_{m+2}, \dots , respectively, the second, third, etc.

Let A_j be the set of all elements of D^* whose first j coefficients belong to Z . We shall prove that each A_j is a normal divisor of the group D^* . Indeed, let $\alpha, \beta \in A_j$. It is easy to see that the element $\alpha\beta$ also belongs to A_j . If $\alpha \in A_j$, then α^{-1} also belongs to A_j , as one verifies by comparing the first j coefficients in the equality $\alpha\alpha^{-1} = 1$ and taking into account that the first j coefficients of the element α belong to the center Z . Now let $\alpha \in A_j$ and let β be an arbitrary element of the group D^* . Then the first j coefficients of the element $\gamma = \beta\alpha\beta^{-1}$ coincide with the corresponding coefficients of the element α , as we verify by comparing the first j coefficients in the equality $\gamma\beta = \beta\alpha$. Consequently, γ also belongs to A_j . Thus, indeed, A_j is a normal divisor of the group D^* .

1. From the definition of the sets A_j it follows that

$$A_1 \subset A_2 \subset A_3 \subset \dots, \quad \bigcap_{j=1}^{\infty} A_j = C^*,$$

where C is the center of the division ring D .

2. We shall prove that $[A_1, A_j] \subseteq A_{j+1}$. This means that for any $a \in A_j$, $\beta \in A_1$, the element $\beta a \beta^{-1} a^{-1} \in A_{j+1}$. As was already mentioned, for $a \in A_j$ and any $\beta \in D^*$, the first j coefficients of the element $\beta a \beta^{-1}$ coincide with the corresponding coefficients of the element a . If, moreover, $\beta \in A_1$, then the corresponding first $j + 1$ coefficients also coincide. Therefore $(\beta a \beta^{-1}) a^{-1}$ has the form $1 + b_{j+1} t^{j+1} + \dots$, i.e. $\beta a \beta^{-1} a^{-1} \in A_{j+1}$, as was required to prove.
3. To complete the proof of the theorem it remains to show that D^*/A_1 is isomorphic to K^*/Z^* . Define a mapping $\varphi : D^* \rightarrow K^*$ as follows. If a is an arbitrary element of D^* and a_m is the first coefficient of the element a , then $a\varphi = a_m$. It is easy to see that the mapping φ is a homomorphism of the group D^* onto the group K^* . The kernel N of this homomorphism consists of all elements of D^* whose first coefficient is equal to 1. Thus $N \subseteq A_1$. Consequently, the homomorphism φ induces a homomorphism of the group A_1 onto Z^* with the same kernel N . Hence $D^*/N \simeq K^*$, $A_1/N \simeq Z^*$. From this,

$$(D^*/N)/(A_1/N) \simeq K^*/Z^*,$$

i.e. $D^*/A_1 \simeq K^*/Z^*$. The theorem is proved.

Remark 1. Each normal divisor A_j , $j = 1, 2, \dots$, of the group D^* is a ZD -group, i.e. A_j has a descending central series. In particular, A_j is an RK -group and, still more, an RN -group, i.e. A_j has a solvable normal system. However, A_j is not contained in the center C^* . It is clear that in the factor group D^*/C^* there exist nontrivial normal divisors A_j/C^* , which are RN -groups (RK -groups, ZD -groups).

Remark 2. From the primarity of the factor group K^*/Z^* (see ⁽¹⁾) it follows that A_1 is a primary normal divisor of the group D^* . Thus, for the multiplicative group of a division ring, the center need not be the only primary normal divisor.

Remark 3. In ⁽²⁾, p. 191, an example is given showing that the multiplicative group of a division ring may contain an infinite descending invariant series. Our theorem shows that such a situation is quite general.

§2. The Cartan-Brauer-Hua theorem (see ⁽²⁾, p. 186) establishes that if M is a proper subdivision ring of a division ring K , not contained in the center Z of the division ring K , then the multiplicative group M^* is not a normal divisor of the group K^* . In ^(3,4) it is shown that in this case the group M^* is not even a member of any finite normal series of the group K^* . Here we give a further generalization of this result.

Theorem 2. *Let M be a proper subdivision ring of a division ring K , not contained in the center Z of the division ring K . Then the multiplicative group M^* is not a member of any ascending (in general, transfinite) normal series of the group K^* .*

As usual, we shall call a subgroup H **subinvariant** in a group G if H is a member of some ascending (in general, transfinite) normal series of the group G . B. I. Plotkin ⁽⁵⁾ proved that if $R(G)$ is the radical of the group G , then for any subinvariant subgroup H of the group G the relation $R(H) = R(G) \cap H$ holds. On the other hand, the author showed ⁽¹⁾ that for the multiplicative group K^* of the division ring K the radical coincides with the center Z^* . Consequently, if M^* is subinvariant in K^* , then the center of the division ring M must lie in the center of the division ring K . Therefore, if M is in fact a field not contained in Z , then M^* cannot be a subinvariant subgroup of the group K^* . Thus the theorem is proved for the case when the subdivision ring M is commutative.

To prove the theorem in the general case, we shall need some lemmas. Denote $V(M) = \{x \in K \mid xm = mx \text{ for all } m \in M\}$ and $N(M) = \{y \in K \mid y^{-1}My = M\}$. $V(M)$ is called the centralizer of the subfield M in K .

Lemma 1 (see ⁽²⁾, p. 186). *Let L and M be subfields of the field K and let $L^* \subseteq N(M)$. Then either $L \subseteq M$, or $L \subseteq V(M)$.*

Lemma 2. *Let M be a noncommutative proper subfield of the field K , and let $V(M)$ be commutative. Then M^* is not a subinvariant subgroup in K^* .*

Proof. Suppose that M^* is subinvariant and $M^* \subset A_1 \subset A_2 \subset \dots \subset A_\alpha \subset A_{\alpha+1} \subset \dots \subset A_\gamma = K^*$ is an ascending normal series passing through M^* . We shall show that every term of this series is contained in $N(M)$. This will mean that also $A_\gamma = K^* \subseteq N(M)$, i.e. M^* is a normal divisor of the group K^* , which contradicts the Cartan-Brauer-Hua theorem.

By construction, $A_1 \subseteq N(M)$. Suppose that for all $\beta < \alpha$, $A_\beta \subseteq N(M)$. If α is

a limit transfinite number, then

$$A_\alpha = \bigcup_{\beta < \alpha} A_\beta \subseteq N(M).$$

If, however, $\alpha - 1$ exists, then for any $x \in A_\alpha$ we have

$$x^{-1}M^*x \subseteq x^{-1}A_{\alpha-1}x = A_{\alpha-1} \subseteq N(M),$$

i.e. $(x^{-1}Mx)^* \subseteq N(M)$. In view of Lemma 1, either $x^{-1}Mx \subseteq M$, or $x^{-1}Mx \subseteq V(M)$. Since $V(M)$ is commutative and M is noncommutative, $x^{-1}Mx \not\subseteq V(M)$. Consequently, $x^{-1}Mx \subseteq M$ for any $x \in A_\alpha$. Since A_α is a group, $x^{-1}Mx = M$ for any $x \in A_\alpha$, i.e. $A_\alpha \subseteq N(M)$. The lemma is proved.

Let us note that if $V(M) \subset M$, then $V(M)$ is commutative. Thus, we have:

Corollary. *Let M be a noncommutative proper subfield of the field K and let $V(M) \subset M$. Then M^* is not a subinvariant subgroup in K^* .*

We formulate one more lemma, which can be verified.

Lemma 3. *Let M be a subfield of the field K and suppose that there exists an element $a \in V(M)$, $a \notin M$. Let Δ be the minimal subfield containing a and M . Then the centralizer of the subfield M in Δ coincides with the center of the field Δ and, consequently, is commutative.*

Proof of Theorem 2. We have already proved the theorem for the case when M is commutative. Let M be a noncommutative proper subfield of the field K . Suppose that M is subinvariant in K^* . By the corollary to Lemma 2, $V(M)$ is not contained in M . Let $a \in V(M)$, $a \notin M$, and let Δ be the minimal subfield of the field K containing M and a . Then M is a proper subfield of the field Δ , and, by Lemma 3, the centralizer of the subfield M in Δ is commutative. By Lemma 2, M^* is not a subinvariant subgroup in the group Δ^* . But M^* , being subinvariant in K^* , must be subinvariant in every subgroup of the group K^* containing M^* . This contradiction proves the theorem.

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