# MATHEMATICS

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## Abstract

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MATHEMATICS

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# ON LINEAR METHODS OF APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS OF TWO VARIABLES

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Let $H^{\omega_1,\omega_2}$ be the class of $2\pi$-periodic functions $f(x,y)$ of two variables satisfying the condition

$$|f(x_1,y_1)-f(x_2,y_2)| \le \omega_1(|x_2-x_1|)+\omega_2(|y_2-y_1|),$$

where $\omega_1(t)$ and $\omega_2(t)$ are certain moduli of continuity, i.e., nonnegative functions of the argument, continuous at zero and vanishing at zero, monotone and semiadditive.

We shall say that the matrix $\{\lambda_{k,l}^{(m,n)}\}$ $(0 \le k \le m+1,\ 0 \le l \le n+1,\ \lambda_{0,0}^{(m,n)} = 1,\ \lambda_{m+1,l}^{(m,n)} = \lambda_{k,n+1}^{(m,n)} = 0)$ defines on the class $H^{\omega_1,\omega_2}$ a linear method $U_{m,n}(\lambda)$, if to each function of the class $H^{\omega_1,\omega_2}$ there is assigned the sequence of trigonometric polynomials

$$U_{m,n}(f;x,y;\lambda) = \sum_{k=0}^{m}\sum_{l=0}^{n} \lambda_{k,l}^{(m,n)} \mu_{k,l}(a_{k,l}\cos kx\cos ly + b_{k,l}\cos kx\sin ly +$$

$$+c_{k,l}\sin kx\cos ly + d_{k,l}\sin kx\sin ly),$$

where

$$\mu_{k,l} = \begin{cases} 1/4, & \text{if } k=l=0; \\ 1/2, & \text{if } k>0,\ l=0;\ l>0,\ k=0; \\ 1, & \text{if } k,l>0; \end{cases}$$

$a_{k,l}, b_{k,l}, c_{k,l}, d_{k,l}$ are the Fourier coefficients of the function $f(x,y)$.

Let, further,

$$\Delta^2_{kk}\lambda^{(m,n)}_{k,l}=\lambda^{(m,n)}_{k,l}-2\lambda^{(m,n)}_{k+1,l}+\lambda^{(m,n)}_{k+2,l},\qquad \Delta^2_{ll}\lambda^{(m,n)}_{k,l}=\lambda^{(m,n)}_{k,l}-2\lambda^{(m,n)}_{k,l+1}+\lambda^{(m,n)}_{k,l+2},$$

$$\Delta^4_{kkll}\lambda^{(m,n)}_{k,l}=\Delta^2_{kk}(\Delta^2_{ll}\lambda^{(m,n)}_{k,l}).$$

In the present note the following is established.

**Theorem.** If the matrices $\{\lambda^{(m,n)}_{k,l}\}$ are such that, for fixed $m,n$,

$$\Delta^2_{kk}\lambda^{(m,n)}_{k,l}\le 0\qquad (0\le k\le m-1,\ 0\le l\le n);$$

$$\Delta^2_{ll}\lambda^{(m,n)}_{k,l}\le 0\qquad (0\le k\le m,\ 0\le l\le n-1);$$

$$\Delta^4_{kkll}\lambda^{(m,n)}_{kl}\ge 0\qquad (0\le k\le m-1,\ 0\le l\le n-1),$$

and the numbers $\lambda^{(m,n)}_{k,l}$ decrease monotonically with respect to $k$ and $l$, then for the linear method $U_{m,n}(\lambda)$ on the class $H^{\omega_1,\omega_2}$ with moduli of continuity $\omega_1(t)$ and $\omega_2(t)$, satisfying the conditions

$$\delta\int_\delta^1\frac{\omega_1(t)}{t^2}\,dt=O[\omega_1(\delta)],\qquad \delta\int_\delta^1\frac{\omega_2(t)}{t^2}\,dt=O[\omega_2(\delta)]\quad(\text{as }\delta\to 0),\quad (1)$$

there is the estimate

$$\sup_{f\in H^{\omega_1,\omega_2}}\left\|f(x,y)-U_{m,n}(f;x,y;\lambda)\right\|_C=\frac{C_{m,n}(\omega_1,\omega_2)}{\pi^2}\sum_{k=1}^m\sum_{l=1}^n\frac{\lambda^{(m,n)}_{k,l}}{(m-k+1)(n-l+1)}$$
$$+O\left\{\left[\omega_1\left(\frac{1}{m}\right)+\omega_2\left(\frac{1}{n}\right)\right]\left[\sum_{k=1}^m\frac{\lambda^{(m,n)}_{k,0}}{m-k+1}+\sum_{l=1}^n\frac{\lambda^{(m,n)}_{0,l}}{n-l+1}\right]\right\}$$
$$+O\left[\omega_1\left(\frac{1}{m}\right)+\omega_2\left(\frac{1}{n}\right)\right],\quad (2)$$

where

$$C_{m,n}(\omega_1,\omega_2)=\sup_{f\in H^{\omega_1,\omega_2}}\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}f(x,y)\cos mx\cos ny\,dx\,dy.$$

The proof of the theorem is based on the following lemma. Let

$$\sigma_{m,n}^{(p,q)}(f;x,y) = \frac{1}{(p+1)(q+1)} \sum_{m-p}^{m} \sum_{n-q}^{n} S_{k,l}(f;x,y),$$

where $S_{k,l}(f;x,y)$ are the partial sums of the Fourier series of the function $f(x,y)$.

**Lemma.** If $\omega_1(t)$ and $\omega_2(t)$ are moduli of continuity satisfying conditions (1), then:

a) uniformly with respect to all $m, n,\ 0 \le p \le m,\ 0 \le q \le n$, the estimate

$$\sup_{f\in H^{\omega_1,\omega_2}} \left\| f(x,y) - \sigma_{m,n}^{(p,q)}(f;x,y) \right\|_C = \frac{C_{m,n}(\omega_1,\omega_2)}{\pi^2} \ln\frac{m}{p+1} \cdot \ln\frac{n}{q+1} + O\left\{\left[\ln\frac{m}{p+1} + \ln\frac{n}{q+1}\right]\left[\omega_1\left(\frac{1}{m}\right) + \omega_2\left(\frac{1}{n}\right)\right]\right\} + O\left[\omega_1\left(\frac{1}{m}\right) + \omega_2\left(\frac{1}{n}\right)\right]; \tag{3}$$

b) the extremal function in estimate (3) depends only on $m$ and $n$, and does not depend on $p$ and $q$.

Since in the case of convex moduli of continuity $\omega_1(t)$ and $\omega_2(t)$ the equality (3)

$$C_{m,n}(\omega_1,\omega_2) = \frac{8}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \min\left\{\omega_1\left(\frac{2x}{m}\right), \omega_2\left(\frac{2y}{n}\right)\right\} \sin x \sin y \, dx\, dy, \tag{4}$$

holds, then for $\omega_1(t) = t^\alpha,\ \omega_2(t) = t^\beta\ (0 < \alpha, \beta < 1)$ we obtain from estimate (2) the result of V. G. Ponomarenko (3).

The first result, similar to estimate (2), belongs to P. T. Bugayets (1) and pertains also to the case of approximation by Fourier sums $\lambda_{k,l}^{(m,n)} = 1$; it can be obtained from estimate (3) for $p = q = 0$, if one takes into account equality (4), as well as the circumstance that estimate (3) under the condition $p \le \theta m,\ q \le \theta n$ $(0 < \theta < 1)$ holds for arbitrary moduli of continuity.

Estimates (2) and (3) are analogues of the corresponding estimates for the one-dimensional case (see (4, 2)).

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*Note: Figure translations are in progress. See original paper for figures.*