CONSTRUCTION AND PROPERTIES OF INVARIANT MEASURABLE PARTITIONS

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Abstract

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CONSTRUCTION AND PROPERTIES OF INVARIANT MEASURABLE PARTITIONS

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1. Terminology and notation.

We shall use the terminology of the survey article (1), which describes the current state and problems of the theory of transformations with invariant measure.

M denotes a Lebesgue space; ε is the partition of the space M into individual points; ν is the trivial partition with the single element M; the notation $\xi \leqslant \eta$, where ξ and η are partitions, means that η is a subpartition of ξ; $\prod$ and $\bigcap$ denote the product and intersection of measurable partitions; $H(\xi)$ is the entropy of the measurable partition ξ; $H(\xi \mid \eta)$ is the mean conditional entropy of the measurable partition ξ with respect to the measurable partition η (for the definition and properties of the function $H(\xi \mid \eta)$, see (2)); Z is the set of partitions ξ with $H(\xi) < \infty$; T is an automorphism of the space M; ξ_T^n (briefly ξ^n), ξ_T^- (briefly ξ^-) and ξ_T are partitions defined from the partition ξ and the automorphism T by the formulas

$$\xi_T^n = \prod_{k=0}^{n-1} T^k \xi, \qquad \xi_T^- = \prod_{k=1}^{\infty} T^{-k} \xi, \qquad \xi_T = \prod_{k=-\infty}^{\infty} T^k \xi;$$

$h(T, \xi)$ and $h(T)$ are functions defined by the formulas

$$h(T, \xi) = H(\xi \mid \xi_T^-), \qquad h(T) = \sup h(T, \xi) \quad (\xi \in Z);$$

$h(T)$ is the entropy of the automorphism T.

M. S. Pinsker showed (3) that every automorphism has a largest factor-automorphism with zero entropy; in other words, for every automorphism T there exists a measurable partition $\pi(T)$ such that, for $\xi \in Z$, the condition $\xi \leqslant \pi(T)$ is equivalent to the equality $h(T, \xi) = 0$. If $\pi(T) = \nu$, then T is called an automorphism with completely positive entropy. T is called a K-automorphism, or a Kolmogorov automorphism, if there exists a measurable partition ζ with three properties: $T\zeta \geqslant \zeta$,

$$\prod_n T^n\zeta = \varepsilon, \qquad \bigcap_n T^n\zeta = \nu.$$

2. Statement of results.

Theorem 1. For every automorphism T there exists a measurable partition ζ with four properties: a) $T\zeta \geqslant \zeta$; b) $\prod_k T^k\zeta = \varepsilon$; c) $\bigcap_k T^k\zeta = \pi(T)$; d) $H(T\zeta/\zeta) = h(T)$.

Theorem 2. If a measurable partition ζ has, with respect to an automorphism T, properties a), b), then $\bigcap_n T^n\zeta \geqslant \pi(T)$. If ζ has properties a), d) and $h(T) < \infty$, then $\bigcap_n T^n\zeta \leqslant \pi(T)$. Thus, if $h(T) < \infty$, then c) follows from a), b), d).

Corollary 1. The class of automorphisms with completely positive entropy coincides with the class of K-automorphisms.

Corollary 2. The factor automorphism of a K-automorphism is a K-automorphism.

Corollary 3. An automorphism inverse to a K-automorphism is a K-automorphism.

Corollary 4. An automorphism generated by a stationary Gaussian sequence with absolutely continuous spectrum is a K-automorphism.

Corollary 5. The unitary operator U conjugate to an automorphism T has, in the orthogonal complement to the subspace of functions with integrable square of the modulus that are constant mod 0 on the elements of the partition $\pi(T)$, a countably multiple Lebesgue spectrum.

Theorems 1 and 2 are proved below. Here we shall derive from them Corollaries 1–5. The fact that a K-automorphism has completely positive entropy was proved by M. S. Pinsker ([3]); this also follows from Theorem 2. The converse proposition follows from Theorem 1. Corollaries 2 and 3 follow in an obvious way from Corollary 1. Corollary 4 follows from Corollary 1 and from the fact that, in the case under consideration, the automorphism has completely positive entropy (see ([3])). Corollary 5 follows from the fact that the orthogonal complement of a unitary subring in a unitary ring is countably infinite-dimensional (see ([5])).

3. Three lemmas. We shall denote measurable partitions by α, β, γ.

Lemma 1. If $\beta \preccurlyeq \alpha$ and $H(\alpha \mid \beta^-) < \infty$, or $\alpha \preccurlyeq \beta$ and $H(\beta \mid \alpha^-) < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n \mid \beta^-) = H(\alpha \mid \alpha^-). \tag{1}$$

Proof. In the first case the sequence of partitions $T^{-n}(\beta^-\alpha^n)$ converges, increasing, to α^-, so that $H(\alpha \mid T^{-n}(\beta^-\alpha^n)) \to H(\alpha \mid \alpha^-)$. Applying the method of arithmetic means and using the relation

$$H(\alpha^n \mid \beta^-) = \sum_{k=1}^{n} H(\alpha \mid T^{-k}(\beta^-\alpha^n)),$$

we obtain (1). In the second case

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n \mid \beta^-) = \lim_{n\to\infty} \left[\frac{1}{n} H(\beta^n \mid \beta^-) - \frac{1}{n} H(\beta^n \mid \alpha^n\beta^-)\right] \geq$$

$$\geq H(\beta \mid \beta^-) - \lim_{n\to\infty} \frac{1}{n} H(\beta^n \mid \alpha^n\alpha^-) = \lim_{n\to\infty} \left[\frac{1}{n} H(\beta^n \mid \alpha^-\alpha^{-1}) - \frac{1}{n} H(\beta^n \mid \alpha^n\alpha^-)\right] =$$

$$= \lim_{n\to\infty} \frac{1}{n} H(\alpha^n \mid \alpha^-) = H(\alpha \mid \alpha^-),$$

and the reverse inequality is obvious.

Lemma 2. If $\alpha \preccurlyeq \beta$ and $H(\beta\gamma \mid \beta^-) < \infty$, then

$$\lim_{n\to\infty} H(\alpha \mid \beta^- T^{-n}\gamma^-) = H(\alpha \mid \beta^-). \tag{2}$$

Proof. Since

$$\sum_{k=0}^{n-1} H(\beta \mid \beta^- T^{-k}\gamma^-) = H(\beta^n \mid \beta^-\gamma^-),$$

for $\alpha = \beta$ formula (2) follows from Lemma 1 and the theorem on arithmetic means. In the general case,

$$\lim_{n\to\infty} H(\alpha \mid \beta^- T^{-n}\gamma^-) = \lim_{n\to\infty} H(\beta \mid \beta^- T^{-n}\gamma^-) - \lim_{n\to\infty} H(\beta \mid \alpha\beta^- T^{-n}\gamma^-) \geq$$

$$\geq H(\beta \mid \beta^-) - H(\beta \mid \alpha\beta^-) = H(\alpha \mid \beta^-),$$

and the reverse inequality is obvious.

Lemma 3 (4). If $H(\alpha\beta \mid \beta^-) < \infty$, then

$$H(\alpha\beta \mid \alpha^-\beta^-) = H(\alpha \mid \alpha^-\beta_T) + H(\beta \mid \beta^-). \tag{3}$$

Proof. Since $H(\alpha \mid \alpha^-\beta^-\beta^n) \to H(\alpha \mid \alpha^-\beta_T)$ and

$$H(\alpha^n \mid \alpha^- \beta^- \beta^n) = \sum_{k=0}^{n-1} H(\alpha \mid \alpha^- \beta^- \beta^k),$$

we have

$$\frac{1}{n} H(\alpha^n \mid \alpha^- \beta^- \beta^n) \to H(\alpha \mid \alpha^- \beta_T).$$

On the other hand, by Lemma 1,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} H(\alpha^n \mid \alpha^- \beta^- \beta^n) &= \lim_{n\to\infty} \left[\frac{1}{n} H(\alpha^n \beta^n \mid \alpha^- \beta^-) - \frac{1}{n} H(\beta^n \mid \alpha^- \beta^-)\right] \\ &= H(\alpha\beta \mid \alpha^- \beta^-) - H(\beta \mid \beta^-).\end{aligned}$$

4. Proof of Theorem 1. Let $\xi_1, \xi_2, \ldots$ be an increasing sequence of partitions from Z, converging to ε, and let $n_1, n_2, \ldots$ be a sequence of integers. Put

$$\eta_p = \prod_{k=1}^{p} T^{-n_k} \xi_k, \qquad \eta = \prod_{k=1}^{\infty} T^{-n_k} \xi_k, \qquad \zeta = \eta^-.$$

It is clear that ζ has properties a) and b). We shall show that if the sequence $n_1, n_2, \ldots$ increases sufficiently rapidly, then

$$\lim_{p\to\infty} \left[H(\eta_p \mid \eta_p^-) - H(\eta_p \mid \zeta)\right] = 0, \tag{4}$$

and that, if this relation is satisfied, then ζ has properties c) and d).

Relation (4) will certainly be satisfied if we subject the choice of the numbers $n_1, n_2, \ldots$ to the conditions

$$H(\eta_p \mid \eta_{q-1}^-) - H(\eta_p \mid \eta_q^-) < \frac{1}{p} \frac{1}{2^{q-p}} \qquad (p < q). \tag{5}$$

Indeed, from (5) it follows that

$$H(\eta_p \mid \eta_p^-) - H(\eta_p \mid \eta^-) < \frac{1}{p}, \qquad H(\eta_p \mid \eta_q^-) - H(\eta_p \mid \zeta) \le \frac{1}{p}.$$

The choice can be made inductively: if the numbers $n_1, \ldots, n_{q-1}$ have been chosen, then n_q is chosen so large that the inequalities (5) are satisfied for $p = 1, \ldots, q-1$. This is possible by Lemma 2.

Since the sequence $\xi_1, \xi_2, \ldots$ increases and converges to ε, we have

$$H(\eta_p \mid \eta_p^-) = h(T, \eta_p) = h(T, \xi_p) \to h(T).$$

Since the sequence $\eta_1, \eta_2, \ldots$ increases and converges to η, we have

$$H(\eta_p \mid \zeta) \to H(\eta \mid \zeta) = H(T\zeta \mid \zeta).$$

Therefore d) follows from (4).

Let us show that c) follows from (4). Let $\alpha \in Z$ and $\alpha \leq \bigcap_n T^n \zeta$. Then $\eta_p^- \alpha_T^- \leq \zeta$. Decomposing $H(\eta_p \alpha \mid \eta_p^- \alpha^-)$ in two ways by formula (3), we obtain

$$h(T, \alpha) = H(\alpha \mid \alpha^-) = H(\alpha \mid \alpha^-(\eta_p)_T) + H(\eta_p \mid \eta_p^-) - H(\eta_p \mid \eta_p^- \alpha_T). \tag{6}$$

Since the sequence $(\eta_1)_T, (\eta_2)_T, \ldots$ increases and converges to ε, we have

$$H(\alpha \mid \alpha^-(\eta_p)_T) \to 0,$$

while the difference

$$H(\eta_p \mid \eta_p^-) - H(\eta_p \mid \eta_p^- \alpha_T)$$

does not exceed the difference (4) and therefore also tends to zero. Consequently,

$$h(T, \alpha) = 0, \qquad \alpha \leq \pi(T)$$

and

$$\bigcap_n T^n \zeta \leq \pi(T).$$

The reverse inequality follows from Theorem 2.

5. Proof of Theorem 2. Put

$$\zeta_0 = \bigcap_n T^n \zeta,$$

and let $\eta \in Z$, $\eta \leq \pi(T)$. Further, let ξ be a partition from Z satisfying, for some m, the inequality $\xi \leq T^m \zeta$. We shall show that

$$H(\xi \mid \zeta_0 \eta_T) = H(\xi \mid \zeta_0).$$

By condition b), this will prove that $\zeta_0 \eta_T = \zeta_0$, i.e. that $\eta \leq \zeta_0$, and hence it will be proved that

$$\pi(T) \leq \zeta_0.$$

For any natural p,

$$H(\xi \mid \zeta_0) \geq H(\xi \mid \zeta_0 \eta_T) \geq H(\xi \mid \xi^-_{\tau^p} \zeta_0 \eta_T) \tag{7}$$

(where $\xi^-_{\tau^p} = \prod_{k=1}^{\infty} T^{-pk}\xi$). It is clear that $T\xi_0 = \xi_0$, and since $\eta \ll \pi(T)$, we have $\eta_T = \eta^- = T^{pn}\eta^-$ (for any n). This makes it possible to apply Lemma 2 to the right-hand side of formula (7), with T^p in the role of T (one must put $\alpha = \xi,\ \beta = \xi\xi_0,\ \gamma = \prod_{k=1}^{p} T^{-kn}\eta$). The result is:

$$H(\xi \mid \xi^-_{\tau^p} \zeta_0 \eta_T) = H(\xi \mid \xi^-_{\tau^p} \zeta_0).$$

Since the sequence of partitions $\xi^-_{\tau^p}\zeta_0$ decreases and converges to ζ_0 as $p \to \infty$, it follows that $H(\xi \mid \xi^-_{\tau^p}\zeta_0) \to H(\xi \mid \zeta_0)$. Consequently,

$$H(\xi \mid \zeta_0 \eta_T) = H(\xi \mid \zeta_0).$$

The proof of the second part of the theorem is close to the arguments of the preceding paragraph. Let $\alpha \in Z$ and $\alpha \ll \bigcap_n T^n\xi$. Choose in Z an increasing sequence $\eta_1, \eta_2, \dots$ converging to ξ, and again write formula (6). Since $\alpha \ll \xi$, we have $H(\alpha \mid \eta_p) \to 0$, and since $H(\alpha \mid \alpha^-(\eta_p)_T) \leq H(\alpha \mid \eta_p)$, we also have $H(\alpha \mid \alpha^-(\eta_p)_T) \to 0$. The difference $H(\eta_p \mid \eta_p^-) - H(\eta_p \mid \eta_p^- \alpha_T)$ does not exceed the difference $h(T) - H(\eta_p \mid \xi^-)$, as $p \to \infty$, which converges to

$$h(T) - H(\xi \mid \xi^-) = h(T) - H(T\xi \mid \xi) = 0.$$

Therefore, $h(T, \alpha) = 0,\ \alpha \ll \pi(T)$, and $\bigcap_n T^n\xi \ll \pi(T)$.

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Note: Figure translations are in progress. See original paper for figures.

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