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Abstract

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MATHEMATICS

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ON MULTIDIMENSIONAL SURFACES CARRYING A NET OF CONJUGATE LINES

(Presented by Academician P. S. Aleksandrov, 3 IV 1961)

1. In this paper we study surfaces V_n of the affine space E_N all of whose asymptotic quadratic forms can be simultaneously reduced to canonical form—surfaces carrying a net of conjugate lines. The question of the structure of such surfaces was considered by V. T. Bazylev in [1]. We study this question in greater detail and show that the structure of the surface V_n is completely determined by the structure of the matrix of eigenvalues of its asymptotic quadratic forms. We then consider the question of the possibility of adjoining to the surface V_n a family of normalizers lying in its osculating planes and inducing on it a connection with respect to which the lines of the conjugate net are geodesics. This question also turns out to be connected with the structure of the matrix of eigenvalues of the asymptotic quadratic forms of the surface V_n .

All constructions of the present paper are projectively invariant. However, for formal reasons it has proved more convenient for us to regard the surface V_n as embedded in the affine space E_N .

2. Let us attach to the surface V_n an affine frame whose point A coincides with the current point of the surface, while the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ lie in its tangent plane and are tangent to the lines of the conjugate net passing through the point A . The infinitesimal displacement of this frame will be determined by the equations

$$dA = \omega^p \mathbf{e}_p + \omega^\alpha \mathbf{e}_\alpha, \quad d\mathbf{e}_p = \omega_p^q \mathbf{e}_q + \omega_p^\alpha \mathbf{e}_\alpha, \quad d\mathbf{e}_\alpha = \omega_\alpha^p \mathbf{e}_p + \omega_\alpha^\beta \mathbf{e}_\beta, \\ (p, q = 1, \dots, n; \quad \alpha, \beta = n + 1, \dots, N), \quad (1)$$

where

$$\omega^\alpha = 0, \quad \omega_p^\alpha = a_p^\alpha \omega^p, \quad \omega_p^q = \sum_s l_{ps}^q \omega^s \quad (p \neq q),$$

and in the latter equations, and everywhere below, summation over the indices p, q, s will be carried out only when this is indicated by the sign \sum . The forms

ω^p will be linearly independent on the surface V_n , and its asymptotic quadratic forms will take the form

$$\varphi^\alpha = \sum_p a_p^\alpha (\omega^p)^2;$$

the p -th conjugate line of the surface V_n is defined by the equations: $\omega^q = 0$ for $q \neq p$, $\omega^p \neq 0$.

The vectors $\mathbf{a}_p = a_p^\alpha \mathbf{e}_\alpha$, together with the vectors \mathbf{e}_p , determine the osculating plane of the surface V_n . Its dimension is $n + n_1$, where $n_1 \leq n$. We assume that the surface V_n is tangentially nondegenerate. This means that none of the vectors \mathbf{a}_p on it vanishes.

Let us place the vectors \mathbf{e}_{n+p_1} of the frame in the osculating plane of the surface V_n , and the vectors \mathbf{e}_{α_1} outside it ($p_1 = 1, \dots, n_1$; $\alpha_1 = n + n_1 + 1, \dots, N$). Then $\mathbf{a}_p = a_p^{n+p_1} \mathbf{e}_{n+p_1}$, and the matrix of the components of these vectors assumes the form:

$$A_n^{n_1} = \left\| \begin{array}{cccc} a_1^{n+1} & a_2^{n+1} & \dots & a_n^{n+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_1^{n+n_1} & a_2^{n+n_1} & \dots & a_n^{n+n_1} \end{array} \right\|. \quad (2)$$

The rank of this matrix is equal to n_1 .

3. We shall say that a surface V_n decomposes into the direct sum of surfaces V_m and V_{n-m} if it decomposes into an $(n-m)$ -parameter family of surfaces V_m and an m -parameter family of surfaces V_{n-m} , in such a way that through each point of the surface V_n there passes one surface of each family, intersecting only at this point.

The following theorems show how the structure of the surface V_n is related to the structure of its matrix $A_n^{n_1}$.

Theorem 1A. If the matrix $A_n^{n_1}$ of the surface V_n can be brought to the form

$$\left\| \begin{array}{cc} A_m^{m_1} & 0 \\ 0 & A_{n-m}^{n_1-m_1} \end{array} \right\|$$

($m_1 \leq m$; $n_1 - m_1 \leq n - m$), then the surface V_n decomposes into the direct sum of surfaces V_m and V_{n-m} , bearing nets of conjugate lines, and its osculating plane will be the direct sum of the osculating planes E_{m+m_1} and $E_{n+n_1-m-m_1}$ of the surfaces V_m and V_{n-m} .

Theorem 1B. For $n_1 = n$ the matrix $A_n^{n_1}$ of the surface V_n can be brought to the form

$$A_n^n = \left\| \begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{array} \right\|$$

and the surface V_n is a manifold of a special projective type, considered by Cartan in paper ⁽²⁾. Such a surface, generally speaking, does not lie entirely in its osculating plane, and the dimension of its second osculating plane is $2n + n_2$, where $n_2 \leq n$.

Theorem 1C. If, for $n_1 < n$, on the surface V_n the rank of each subsystem of the system of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$, consisting of $n - 1$ vectors, is equal to n_1 , then the surface V_n lies entirely in its osculating plane E_{n+n_1} . In this case $N = n + n_1$.

Theorem 1D. If, for $n_1 < n$, the conditions of Theorem 1C are not satisfied on the surface V_n , then its matrix $A_n^{n_1}$ can be brought to the form

$$\left\| \begin{array}{cc} A_m^{m_1} & 0 \\ 0 & A_{n-m}^{n-m} \end{array} \right\|, \quad (3)$$

where $m_1 = n_1 - (n - m)$, and the matrix $A_m^{m_1}$ satisfies the condition of Theorem 1C. The surface V_n then decomposes into the direct sum of surfaces V_m , lying entirely in their osculating planes E_{m+m_1} , and Cartan surfaces V_{n-m} , the dimension of whose osculating planes is equal to $2(n - m)$. The dimension of the second osculating plane of the surface V_n is equal to $n + n_1 + n_2$, where $n_2 \leq n - m$. In this case the planes E_{m+m_1} describe a surface of rank $n - m$ of the special projective type ⁽²⁾, the dimension of whose tangent planes is $m + n_1$, and the osculating planes coincide with the osculating planes of the surface V_n .

4. We say that a conjugate net of a surface V_n is **completely holonomic** if each of the equations

$$\omega^p = 0,$$

which determine the sub-surfaces of the net, is completely integrable. If, however, no subsystem consisting of fewer than $n - 1$ such equations is completely integrable, then we call the conjugate net of the surface V_n **irreducible**. The condition for complete holonomicity of the conjugate net of the surface V_n is the vanishing of the coefficients l_{ps}^q in the expansion (1) of the forms ω_p^q for $p \neq q$, $p \neq s$, $q \neq s$.

The following theorems hold:

Theorem 2A. If no three of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of the surface V_n lie in one plane, then the net of conjugate lines of this surface will be completely holonomic.

Theorem 2B. An irreducible net of conjugate lines can be carried only by surfaces V_n of class 2 whose system of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ is such that no two of them are collinear, and by surfaces of class 1.

Theorem 2V. The surface V_n may decompose into an $(n-m)$ -parameter family of surfaces V_m carrying an irreducible net of conjugate lines in two cases:

- a) when, in the system of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of this surface, m vectors ($m \geq 3$) lie in one plane which contains none of the remaining vectors of the system, while no two of them are collinear, and the focal planes of the plane generating surfaces $S_{n-m,n}$ of rank m ⁽³⁾, tangent to the surface V_n at the points of its sub-surfaces V_m , belong to one pencil; the class of the surfaces V_m in this case is equal to two;
 - b) when, in the system of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of the surface V_n , m vectors are collinear; in this case the class of the surfaces V_m is equal to one.
5. Let E_{n_1} be a normalizer of the surface V_n , passing through its point A and lying in its osculating plane E_{n+n_1} . We shall call this normalizer a **Foss normal** of the surface V_n if it is intersected by all two-dimensional osculating planes of the conjugate lines of this surface passing through the point A . With respect to the affine connection induced on V_n by its Foss normals, the lines of the conjugate net of this surface will be geodesic lines.

The following theorems show how the question of the existence and construction of a Foss normal of the surface V_n is connected with the structure of its matrix $A_n^{n_1}$.

Theorem 3A. If the condition of Theorem 1A is fulfilled on the surface V_n and it has a Foss normal, then this normal will be the direct sum of the Foss normals of the surfaces V_m and V_{n-m} into which the surface V_n decomposes.

Theorem 3B. If the surface V_n is a Cartan surface, then at each of its points it has a family of Foss normals depending on n parameters.

Theorem 3V. If, for $n_1 = n - 1$, the matrix $A_n^{n_1}$ satisfies the condition of Theorem 1B, then the surface V_n has at each of its points a unique Foss normal.

Theorem 3G. If, for $n_1 = n - 1$, the matrix $A_n^{n_1}$ has the form (3), then the surface V_n has at each of its points a family of Foss normals depending on $n - m$ parameters.

Theorem 3D. For $n_1 < n - 1$, the surface V_n , generally speaking, has no Foss normal. It has a Foss normal only in the case when the coefficients l_{qq}^p in the expansion (1) of the forms ω_q^p of this surface can be made zero.

6. Consider a surface V_n for $n_1 < n - 1$. Suppose that the net of its conjugate lines is completely holonomic and that at each of its points it has a Foss normal. Then the following theorems hold:

Theorem 4A. If, on the surface V_n , the rank of each subsystem of the system of vectors a_1, \dots, a_n consisting of $n-2$ vectors is equal to n_1 , then the congruence of Fosse normals of this surface decomposes into n families of developable surfaces, and these developable surfaces correspond to the conjugate lines of the surface V_n . Each conjugate line of the surface V_n and the corresponding developable surface of the congruence of Fosse normals lie entirely in one (n_1+1) -dimensional plane. Such surfaces V_n exist with arbitrariness $s_1 = (2n_1 + n - 1)n$ functions of one argument.

An example of such a surface is the translation surface

$$r(u^1, \dots, u^n) = r_1(u^1) + \dots + r_n(u^n),$$

the conjugate lines $r_p(u^p)$ of which belong entirely to their own $(n_1 + 1)$ -dimensional osculating planes, while all these osculating planes belong to one pencil with an n_1 -dimensional axis.

Theorem 4B. If, on the surface V_n , the condition of Theorem 4A is not satisfied, but the condition of Theorem 1B is satisfied, then its matrix $A_n^{n_1}$ can be reduced to the form

$$\left\| \begin{array}{cc} A_m^{m_1} & A_{n-m}^{m_1} \\ 0 & A_{n-m}^{n_1-m_1} \end{array} \right\|,$$

where $n_1 - m_1 = n - m - 1$, and the matrix $A_{n-m}^{n_1-m_1}$ satisfies the condition of Theorem 3B. In this case the following cases may occur:

- a) the matrix $A_n^{m_1} = \|A_m^{m_1}, A_{n-m}^{m_1}\|$ satisfies the conditions of Theorem 4A; such surfaces exist with arbitrariness $s_2 = (n - m)(n - m - 1)$ functions of two arguments;
- b) $A_{n-m}^{m_1} = 0$, $m_1 = m - 1$, and the matrix $A_m^{m_1}$ satisfies the condition of Theorem 3B; such a surface V_n decomposes into the direct sum of surfaces V_m and V_{n-m} of the type of Theorem 3B and exists with arbitrariness $s_2 = m(m - 1) + (n - m)(n - m - 1)$ functions of two arguments;
- c) if the conditions of the two preceding items are not satisfied, then the process of reducing the matrix $A_n^{n_1}$ should be continued.

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