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Abstract

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Integral Equations for Periodic Solutions of Whittaker' s Equation

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The paper considers the problem of constructing integral equations for periodic solutions (with period π or 2π) of Whittaker' s equation

$$\frac{d^2u}{dx^2} + \left[A - (p+1)l\cos 2x + \frac{1}{8}l^2\cos 4x\right]u = 0. \tag{1}$$

Equation (1) has periodic solutions for arbitrary values of the parameters p and l and with a proper choice of the constant A. For such solutions one can construct integral equations of the form

$$u(x) = \lambda\int_{-\pi}^{\pi} K(x,y)u(y)\,dy, \tag{2}$$

whose kernels satisfy the equation

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial y^2} + \left[-(p+1)l(\cos 2x - \cos 2y) + \frac{1}{8}l^2(\cos 4x - \cos 4y)\right]K = 0. \tag{3}$$

Explicit expressions for the kernel $K(x,y)$ were obtained in papers ($^{1-3}$). In the present paper two new types of kernels are considered, expressed in terms of a degenerate hypergeometric function, and, in the case of integral p, in terms of Hermite and Laguerre polynomials, and including as special cases all kernels from ($^{1-3}$). In the limit the kernels considered pass into the known kernels for Mathieu functions.

1. We shall find kernels $K(x,y)$ from equation (3). In constructing solutions of this equation we shall be guided by the properties of the integrals of equation (1), for which the sought kernels are intended. Introduce into equation (3) a new unknown function $g(x,y)$ by the formula

$$K = e^{-\frac{1}{4}l(\cos 2x + \cos 2y)}g. \tag{4}$$

For this function the equation assumes the form

$$\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} + l\left(\sin 2x\frac{\partial g}{\partial x} - \sin 2y\frac{\partial g}{\partial y}\right) - pl(\cos 2x - \cos 2y)g = 0. \tag{5}$$

We shall consider two types of solutions of the last equation. First we shall seek the function g in the form

$$g = \varphi(\cos x\cos y)\psi(\sin x\sin y). \tag{6}$$

Then for the functions φ and ψ ordinary differential equations are obtained, whose integrals can be represented in terms of degenerate-

nuent hypergeometric function $F(\alpha,\gamma;z)$. Taking into account the period and parity of the solutions of equation (1), we obtain for them the following kernels:

for even integrals with period π

$$K = e^{-\frac{1}{4}l(\cos 2x+\cos 2y)}\times$$

$$\times F\left(-\frac{\nu}{2}, \frac{1}{2}; l\cos^2 x\cos^2 y\right) F\left(\frac{\nu-p}{2}, \frac{1}{2}; -l\sin^2 x\sin y\right); \tag{7}$$

for even integrals with period 2π

$$K = e^{-\frac{1}{4}l(\cos 2x+\cos 2y)}\cos x\cos y\times$$

$$\times F\left(\frac{1-\nu}{2}, \frac{3}{2}; l\cos^2 x\cos^2 y\right) F\left(\frac{\nu-p}{2}, \frac{1}{2}; -l\sin^2 x\sin^2 y\right); \tag{8}$$

for odd integrals with period π

$$K = e^{-\frac{1}{4}l(\cos 2x+\cos 2y)}\sin 2x\sin 2y\times$$

$$\times F\left(\frac{1-\nu}{2}, \frac{3}{2}; l\cos^2 x\cos^2 y\right) F\left(\frac{\nu-p+1}{2}, \frac{3}{2}; -l\sin^2 x\sin^2 y\right); \tag{9}$$

for odd integrals with period 2π

$$K = e^{-\frac{1}{4}l(\cos 2x+\cos 2y)}\sin x\sin y\times$$

$$\times F\left(-\frac{\nu}{2}, \frac{1}{2}; l\cos^2 x\cos^2 y\right) F\left(\frac{\nu-p+1}{2}, \frac{3}{2}; -l\sin^2 x\sin^2 y\right) \tag{10}$$

(ν is an arbitrary parameter).

If in formula (7) we put $\nu = 0$ and $\nu = p$, in formula (8) $\nu = 1$ and $\nu = p$, in formula (9) $\nu = 1$ and $\nu = p - 1$, and in formula (10) $\nu = 0$ and $\nu = p - 1$, we obtain the kernels from (3).

2. To obtain the second type of kernels, put in equation (5)

$$g = \cos^k(x - y)\,\varphi(\cos 2x + \cos 2y), \qquad k = 0, 1, 2, \ldots . \tag{11}$$

The equation for φ that is obtained in this case is also integrated with the aid of the confluent hypergeometric function. Thus we obtain the kernels

$$K = e^{-\frac{1}{4}l(\cos 2x + \cos 2y)} \cos^k(x - y)\, F\left(\frac{k-p}{2},\ k+1;\ \frac{1}{2}l(\cos 2x + \cos 2y)\right),$$

$$k = 0, 1, 2, \ldots, \tag{12}$$

which correspond to integrals of equation (1) having period π when k is even and to integrals with period 2π when k is odd.

If one takes $k = p$ (with p a positive integer), formula (12) gives Whittaker' s kernel (^{1}). The kernels from (2) are obtained for $k = 0, 1, 2^*$.

3. The case of positive integral p is of special interest. In this case there exist $p + 1$ values of the constant A for which differential equation (1) has a periodic integral expressible in finite form through elementary functions (^{4}). For the integral equation (2), in this case the same number $(p + 1)$ of degenerate kernels can be constructed**, which follows directly from formulas (7)–(10) and (12), in which the parameters ν and k can be chosen so that the hypergeometric functions turn into polynomials. For example, if $p = 2n$, $n = 0, 1, 2, \ldots$, setting in formula (7) $\nu = 2m$; $m = 0, 1, 2, \ldots, n$, we obtain the kernel

$$K = e^{-\frac{1}{4}l(\cos 2x + \cos 2y)} H_{2m}(\sqrt{l}\cos x \cos y)\, H_{2(n-m)}(\sqrt{-l}\sin x \sin y), \tag{13}$$

* More precisely, for $k = 1$ the kernel (12) is a linear combination of the kernels (16), II and (16), III from (^{2})

** In the case of formula (12), the number of degenerate kernels is p+1, if kernels suitable for both even and where $H_n(z)$ is a Hermite polynomial. Analogous results are obtained from formulas (8)–(10). The degenerate case for kernels of type (12) corresponds to the choice $k = p - 2n,\ n = 0, 1, 2, \dots, [p/2]$ for $p = 0, 1, 2, \dots$. The kernels obtained in this case are expressed in terms of Laguerre polynomials $L_n^{\alpha}(z)$:

$$K = e^{-\frac{1}{4}l(\cos 2x + \cos 2y)} \cos^{p-2n}(x - y) L_n^{p-2n}({}^1/_2\, l(\cos 2x + \cos 2y)). \tag{14}$$

4. If l is simultaneously made to tend to zero and p to infinity in such a way that pl tends to a finite limit, which we denote by $2q$, equation (1) goes over into Mathieu' s equation

$$\frac{d^2u}{dx^2} + (A - 2q\cos 2x)u = 0. \tag{15}$$

Under this limiting transition formulas (7)–(10) give, respectively, the kernels

$$\operatorname{ch}(2\sqrt{q}\sin x \sin y), \qquad \cos x \cos y\ \operatorname{ch}(2\sqrt{q}\sin x \sin y),$$

$$\cos x \cos y\ \operatorname{sh}(2\sqrt{q}\sin x \sin y), \qquad \operatorname{sh}(2\sqrt{q}\sin x \sin y). \tag{16}$$

If, however, in formulas (7) and (8) we put $\nu = p$, and in formulas (9) and (10) $\nu = p - 1$, and then carry out the indicated limiting transition, we obtain, respectively,

$$\cos(2\sqrt{q}\cos x \cos y), \qquad \sin(2\sqrt{q}\cos x \cos y),$$

$$\sin x \sin y\ \sin(2\sqrt{q}\cos x \cos y), \qquad \sin x \sin y\ \cos(2\sqrt{q}\cos x \cos y). \tag{17}$$

Finally, from the kernel (12) in the limit we obtain

$$e^{im\varphi} J_m(\sqrt{2q(\cos 2x + \cos 2y)}), \qquad \text{where } \cos\varphi = \frac{\sqrt{2}\cos x \cos y}{\sqrt{\cos 2x + \cos 2y}} \tag{18}$$

and $J_m(z)$ is a Bessel function. The functions (16)–(18) are the known kernels for Mathieu functions.

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CITED LITERATURE

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