The Real and Imaginary Parts of an Unbounded Operator

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Abstract

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The Real and Imaginary Parts of an Unbounded Operator

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Every bounded operator B admits a decomposition into real and imaginary parts

$$B = \frac{B + B^*}{2} + \frac{i}{2}\frac{B - B^*}{i}.$$

Such a decomposition plays an important role in the spectral analysis of bounded non-self-adjoint operators ($^{1-6}$). However, if the operator B is unbounded, then the decomposition loses its meaning, since the operators B and B^* have, generally speaking, different domains of definition. In the present paper it is shown that, using the concept of generalized elements of a Hilbert space, one can also obtain an analogous representation for unbounded operators.

1. Let an unbounded operator T with dense domain of definition D_T be given in a Hilbert space H. Suppose also that on the set $D_T \cap D_{T^*}$ the operator T is symmetric. Define on the set $D = D_T + D_{T^*}$ the operator

$$\tilde{T}f = Tf_1 + T^*f_2 \quad (f = f_1 + f_2,\ f_1 \in D_T,\ f_2 \in D_{T^*}). \tag{1}$$

The operator $\tilde{T}$ is, obviously, linear. Note that if $f \in D_T$, $g \in D_{T^*}$, then $\tilde{T}f = Tf$, $\tilde{T}g = T^*g$. Introduce in D the new norm $\|f\|_1 = \|\tilde{T}f\| + \|f\|$. It is not difficult to see that the norm introduced satisfies all axioms of a norm and that the new norm defines in D a topology which majorizes the original topology of the space H. The set D with the norm $\|\cdot\|$ will be called the **manifold of basic elements**, and its elements—**basic**. We shall say that every continuous functional $\hat{S}(f)$ on the manifold of basic elements is generated by a generalized element $\hat{S}$, and we shall write $(f, \hat{S})$ for the value of the functional on the basic element $f \in D$. It is easy to see that every element $S \in H$ defines, by

$$(\tilde{T}f, S) = (f, \hat{S}) \tag{2}$$

a continuous functional on the manifold of basic elements, generated by a generalized element. The definition of generalized elements given here is a natural extension of the construction of generalized functions in the sense of S. L. Sobolev ([8]). We shall consider only generalized elements $\hat{S}$ generated by equality (2).

2. We shall say that a linear bounded operator B belongs to the class $(i\Omega)$ ([2]) if its non-Hermitian subspace $H_\theta = \dfrac{B - B^*}{i} H$ is finite-dimensional. A linear unbounded operator T will be assigned to the class Ω_0 if it has an inverse operator T^{-1} belonging to $(i\Omega)$.

If the operator T belongs to the class Ω_0, then the equalities

$$\frac{1}{i}[(Tf_1, g_1) - (f_1, Tg_1)] = \sum_{\alpha,\beta=1}^{r_0} (f_1, \hat{e}_\alpha)\, j'_{\alpha,\beta}\, (\hat{e}_\beta, g_1),$$

$$\frac{1}{i}[(f_2, T^* g_2) - (T^* f_2, g_2)] = \sum_{\alpha,\beta=1}^{r_0} (f_2, \hat{e}_\alpha)\, j'_{\alpha,\beta}\, (\hat{e}_\beta, g_2)$$

$$(f_1, g_1 \in D_T;\ \ f_2, g_2 \in D_{T^*}), \tag{3}$$

hold.

where $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{r_0}$ are generalized elements of the manifold D, J' is a Hermitian matrix satisfying the condition ${J'}^2 = I$.

Indeed, for any $g_1 \in D_T,\ g_2 \in D_{T^*}$ there exist ψ_1 and ψ_2 such that $g_1 = T^{-1}\psi_1,\ g_2 = T^{-*}\psi_2$. Next denote $\varphi_1 = Tf_1,\ \varphi_2 = T^* f_2$. Then

$$\frac{1}{i}[(Tf_1, g_1) - (f_1, Tg_1)] = -\left(\frac{T^{-1} - T^{-1*}}{i}\varphi_1, \psi_1\right),$$

$$\frac{1}{i}[(f_2, T^* g_2) - (T^* f_2, g_2)] = -\left(\frac{T^{-1} - T^{-1*}}{i}\varphi_2, \psi_2\right). \tag{4}$$

Since $T \in \Omega_0$, we have*

$$\frac{T^{-1} - T^{-1*}}{i} f = \sum_{\alpha,\beta=1}^{r_0} (f, e_\alpha) j_{\alpha\beta} e_\beta \qquad (f \in H,\ J^2 = I), \tag{5}$$

By virtue of (1) and (5),

$$-\left(\frac{T^{-1} - T^{-1*}}{i}\varphi_1, \psi_1\right) = \sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha) j'_{\alpha,\beta} (e_\beta, \psi_1) = \sum_{\alpha,\beta=1}^{r_0} (Tf_1, e_\alpha) j'_{\alpha,\beta} (e_\beta, Tg_1) =$$

$$= \sum_{\alpha,\beta=1}^{r_0} (f_1, \hat{e}_\alpha) j'_{\alpha,\beta} (\hat{e}_\beta, g_1), \tag{6}$$

$$-\left(\frac{T^{-1} - T^{-1*}}{i} \varphi_2, \psi_2\right) = \sum_{\alpha,\beta=1}^{r_0} (\varphi_2, e_\alpha) j'_{\alpha,\beta} (e_\beta, \psi_2) =$$

$$= \sum_{\alpha,\beta=1}^{r_0} (T^* f_2, e_\alpha) j'_{\alpha,\beta} (e_\beta, T^* g_2) = \sum_{\alpha,\beta=1}^{r_0} (f_2, \hat{e}_\alpha) j'_{\alpha\beta} (\hat{e}_\beta, g_2).$$

Here $J' = -J$. Equalities (4) and (6) prove our assertion. The generalized vectors $\hat{e}_1, \dots, \hat{e}_{r_0}$ will be called **channel vectors** for the operator T.

3. In connection with the fact that the equality

$$\left(\hat{e}_\alpha, \frac{T^{-1} + T^{-1*}}{2} g\right) = (e_\alpha, g)$$

holds ($\alpha = 1, 2, \dots, r_0$), we extend the definition of the operator

$$\frac{T^{-1} + T^{-1*}}{2}$$

to the generalized channel vectors $\hat{e}_1, \dots, \hat{e}_{r_0}$, setting

$$\frac{T^{-1} + T^{-1*}}{2} \hat{e}_\alpha = e_\alpha \qquad (\alpha = 1, 2, \dots, r_0). \tag{7}$$

At the same time we shall also say that

$$\hat{e}_\alpha = \left(\frac{T^{-1} + T^{-1*}}{2}\right)^{-1} e_\alpha.$$

It is not difficult to see that

$$\left(\frac{T^{-1} + T^{-1*}}{2} \hat{e}_\alpha, \hat{e}_\beta\right) = 0 \qquad (\alpha, \beta = 1, 2, \dots, r_0). \tag{8}$$

Indeed, by virtue of (5),

$$e_\alpha = \sum_{\beta=1}^{r_0} u_{\alpha\beta} g_\beta = \sum_{\beta=1}^{r_0} u_{\alpha\beta} \frac{1}{\omega_\beta} \frac{T^{-1} - T^{-1*}}{i} g_\beta \qquad (\alpha = 1, 2, \dots, r_0), \tag{9}$$

* It is known (^{4}) that if the operator $B \in (i\Omega)$, then

$$\frac{B - B^*}{i} f = \sum_{\alpha,\beta=1}^{r_0} (f, e_\alpha) j_{\alpha\beta} e_\beta,$$

where e_β ($\beta = 1, 2, \dots, r_0$) is a linear combination of the eigenvectors of the operator

$$\frac{B - B^*}{i},$$

corresponding to nonzero eigenvalues, and J is a Hermitian matrix satisfying the condition $J^2 = I$. The vectors $e_1, e_2, \dots, e_{r_0}$ are then called channel vectors.

where $g_1, g_2, \dots, g_{r_0}$ are eigenvectors of the operator $\dfrac{T^{-1} - T^{-1*}}{i}$ in the non-Hermitian subspace H_0, corresponding to the nonzero eigenvalues $\omega_1, \dots, \omega_{r_0}$. Equality (8) now follows easily from (1), (2), (7), and (9). In what follows we shall need the following relations: if $f_1 \in D_T$ and $f_2 \in D_{T^*}$, then

$$\begin{aligned} f_1 = T^{-1}\varphi_1 &= \frac{T^{-1} + T^{-1*}}{2}\varphi_1 + \frac{i}{2}\frac{T^{-1} - T^{-1*}}{i}\varphi_1 \\ &= A^{-1}\varphi_1 + \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha)\, j_{\alpha,\beta} e_\beta \qquad \left(A^{-1} = \frac{T^{-1} + T^{-1*}}{2}\right), \end{aligned} \tag{10}$$

$$f_2 = T^{-1*}\varphi_2 = \frac{T^{-1} + T^{-1*}}{2}\varphi_2 - \frac{i}{2}\frac{T^{-1} - T^{-1*}}{i}\varphi_2 = A^{-1}\varphi_2 - \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0} (\varphi_2, e_\alpha)\, j_{\alpha,\beta} e_\beta.$$

Since the operator A^{-1} is defined on the Hilbert space H and on the vectors $\hat{e}_\alpha$ ($\alpha = 1, 2, \dots, r_0$), the operator will be defined on the domain of definition D_A and on the vectors e_α ($\alpha = 1, 2, \dots, r_0$). We further extend the operator

$$A = \left(\frac{T^{-1} + T^{-1*}}{2}\right)^{-1}$$

by linearity to the sets D_T and D_{T^*}, putting, for arbitrary $f_1 \in D_T$, $f_2 \in D_{T^*}$,

$$\begin{aligned} A_1 f_1 &= AA^{-1}\varphi_1 + \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha)\, j_{\alpha,\beta} A e_\beta \\ &= \varphi_1 + \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha)\, j_{\alpha,\beta} \hat{e}_\beta, \end{aligned} \tag{11}$$

$$A_2 f_2 = AA^{-1}\varphi_2 - \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0}(\varphi_2, e_\alpha)\, j_{\alpha,\beta} A e_\beta$$

$$= \varphi_2 - \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0}(\varphi_2, e_\alpha)\, j_{\alpha,\beta}\hat{e}_\beta.$$

By direct verification one can see that, under such an extension, the property of symmetry is not violated, i.e., for arbitrary $f_1, g_1 \in D_T,\ f_2, g_2 \in D_{T^*}$,

$$(A_1 f_1, g_1) = (f_1, A_1 g_1), \qquad (A_2 f_2, g_2) = (f_2, A_2 g_2).$$

Let us now define on the set $D = D_T + D_{T^*}$ the operator

$$\tilde{A} f = A_1 f_1 + A_2 f_2 \qquad (f \in D,\ f = f_1 + f_2,\ f_1 \in D_T,\ f_2 \in D_{T^*}). \tag{12}$$

We note here that the generalized vector $\tilde{A} f$ does not depend on the representation of the vector f in the form $f = f_1 + f_2$, where $f_1 \in D_T,\ f_2 \in D_{T^*}$. Using equalities (10), (11), (12), it is easy to prove the following assertion. If $f \in D_A$, then $\tilde{A} f = Af$. We shall call the linear operator $\tilde{A}$ a generalized extension of the operator A to the set D. The operator $\tilde{A}$ has the special feature that it may carry elements of the Hilbert space either into elements of the same space or into generalized elements. By verification one can see that the generalized extension $\tilde{A}$ has the property of symmetry, i.e., for arbitrary $f, g \in D$,

$$(\tilde{A} f, g) = (f, \tilde{A} g).$$

We also note that if $f \in D_T,\ g \in D_{T^*}$, then

$$\tilde{A} f = A_1 f, \qquad \tilde{A} g = A_2 g. \tag{13}$$

Main theorem. *If the operator T belongs to the class Ω_0, then the following equalities hold*

$$T f_1 = \tilde{A} f_1 + \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0}(f_1, \hat{e}_\alpha)\, j'_{\alpha,\beta}\hat{e}_\beta,$$

$$T^* f_2 = \tilde{A} f_2 - \frac{i}{2}\sum_{\alpha,\beta=1}^{r_0}(f_2, \hat{e}_\alpha)\, j'_{\alpha,\beta}\hat{e}_\beta \qquad (f_1 \in D_T,\ f_2 \in D_{T^*}), \tag{14}$$

where $\tilde{A}$ is the generalized extension of the operator

$$A = \left(\frac{T^{-1} + T^{-1*}}{2} \right)^{-1}.$$

Proof. Any vectors $f_1 \in D_T$, $f_2 \in D_{T^*}$ can be represented in the form (10). Taking (13) into account, we apply the operator $\widetilde{A}$ to (10):

$$\widetilde{A}f_1 = \varphi_1 + \frac{i}{2} \sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha) j_{\alpha\beta} \hat{e}_\beta, \qquad \widetilde{A}f_2 = \varphi_2 - \frac{i}{2} \sum_{\alpha,\beta=1}^{r_0} (\varphi_2, e_\alpha) j_{\alpha\beta} \hat{e}_\beta. \quad (15)$$

By virtue of (7), (8), (10),

$$(f_1, \hat{e}_\gamma) = (A^{-1}\varphi_1, \hat{e}_\gamma) + \frac{i}{2} \left(\sum_{\alpha,\beta=1}^{r_0} (\varphi_1, e_\alpha) j_{\alpha\beta} e_\beta, \hat{e}_\gamma \right) = (A^{-1}\varphi_1, \hat{e}_\gamma) = (\varphi_1, e_\gamma),$$

$$(f_2, \hat{e}_\gamma) = (A^{-1}\varphi_2, \hat{e}_\gamma) - \frac{i}{2} \left(\sum_{\alpha,\beta=1}^{r_0} (\varphi_2, e_\alpha) j_{\alpha\beta} e_\beta, \hat{e}_\gamma \right) = (A^{-1}\varphi_2, \hat{e}_\gamma) = (\varphi_2, e_\gamma).$$

$$(\gamma = 1, 2, \dots, r_0) \quad (16)$$

Substituting the equalities (16) into (15), we obtain the assertion of the theorem. Let us also note that, by virtue of (3) and (14),

$$\frac{1}{i}[(Tf_1, g_1) + (f, Tg_1)] = (\widetilde{A}f_1, g_1), \quad \frac{1}{i}[(f_2, T^*g_2) + (T^*f_2, g_2)] = (\widetilde{A}f_2, g_2). \quad (17)$$

Taking into account the equalities (3) and (17), the operator

$$\widetilde{B}f = \sum_{\alpha,\beta}^{r_0} (f, \hat{e}_\alpha) j'_{\alpha\beta} \hat{e}_\beta \quad (f \in D).$$

It is natural to call $\widetilde{B}$ the imaginary part of the operator T, and the operator $\widetilde{A}$ the real part. As an example, consider the differentiation operator

$$Tf = \frac{1}{i} \frac{df}{dx} \quad (0 \le x \le e),$$

defined on all absolutely continuous functions $f(x)$ satisfying the boundary condition $f(0) = 0$. The operator T is a quasi-self-adjoint extension (1) of a symmetric operator, i.e., on the set $D_T \cap D_{T^*}$ the operator T is symmetric. Moreover, it is not difficult to see that T^{-1} has one channel vector $e \equiv -i$. Consequently, T belongs to the class Ω_0. The set $D = D_T + D_{T^*}$ consists, obviously, of all absolutely continuous functions. It is easy to see that

$$\widetilde{T} f = T f_1 + T^* f_2 = \frac{1}{i} \frac{df}{dx} \quad (f = f_1 + f_2,\ f \in D).$$

We now define the generalized channel vector of the operator T. By virtue of

$$(3) \quad (f, \hat{e}) = \int_0^e \frac{1}{i} \frac{df}{dx}\, i\, dx = f(e) - f(0).$$

The generalized element $\tilde{e}$, generated by this functional, may be regarded as the difference of δ-functions, i.e. $\hat{e}(x) = \delta(x - e) - \delta(x)$. The real and imaginary parts of the operator T have the form

$$\widetilde{A} f = \frac{1}{i} \frac{df}{dx} + \frac{i}{2}[f(0) + f(e)][\delta(x - e) - \delta(x)], \quad \widetilde{B} f = -(f, \hat{e})\hat{e} = [f(0) -$$

$$- f(e)][\delta(x - e) - \delta(x)]. \quad (f \in D).$$

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Note: Figure translations are in progress. See original paper for figures.

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