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Abstract

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MATHEMATICS

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ON THE DEPENDENCE OF THE SOLUTION OF A MIXED PROBLEM FOR A PARABOLIC EQUATION ON THE BOUNDARY

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In this note we consider the solution of a mixed problem for a linear parabolic equation with two independent variables x and t, satisfying, on the lateral boundary of the domain, one of the three standard boundary conditions. The dependence of the solution on changes in the curves specifying the lateral boundary is investigated. It turns out that if the boundary curves satisfy a Hölder condition with exponent $> 1/2$ and are rectifiable, and the boundary conditions on each of the admissible curves are not changed, then the solution, together with its derivative with respect to x, depends continuously (in the C-metric) on the change of the boundary curve (in the Hölder metric). The coefficients of the parabolic equation may have discontinuities of the first kind on a finite number of curves, along which certain conjugation conditions (see (2)) are prescribed; in this case the solution, together with its first derivative with respect to x, also depends continuously on the lines of discontinuity (if they are taken from the class of rectifiable Hölder curves).

I. Let us first consider the solution $u(x,t)$ of the classical mixed problem for an equation of parabolic type

$$\frac{\partial^2 u}{\partial x^2} = a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u + f(x,t),$$

$$X_1(t) < x < X_2(t), \qquad 0 < t < T, \tag{1}$$

satisfying the initial condition

$$u(x,0) = F(x), \qquad X_1(0) \leqslant x \leqslant X_2(0) \tag{2}$$

and boundary conditions of one of the following types:

$$u(X_i(t), t) = \mu_i(t), \qquad 0 \leqslant t \leqslant T, \qquad i = 1, 2; \tag{3a}$$

$$\frac{\partial u(X_i(t), t)}{\partial x} + \lambda_i(t)\, u(X_i(t), t) = \psi_i(t), \qquad 0 \leqslant t \leqslant T, \qquad i = 1, 2, \tag{3b}$$

provided the compatibility conditions are fulfilled

$$F(X_i(0)) = \mu_i(0) \tag{4a}$$

or

$$F'(X_i(0)) + \lambda_i(0) F(X_i(0)) = \psi_i(0), \qquad i = 1, 2. \tag{4b}$$

We shall call H_α the class of all continuous curves $x = h(t)$ $(0 \leqslant t \leqslant T)$ for which $|h(t)| \leqslant M$, $0 \leqslant t \leqslant T$, and

$$|h(t_1) - h(t_2)| \leqslant K |t_1 - t_2|^{\frac{1+\alpha}{2}}, \qquad t_1, t_2 \in [0, T],$$

where M, K, and α are constants $(0 < \alpha \leqslant 1)$. By V we denote the set of all rectifiable curves from H_α.

Let

$$|h|_t = \sup_{0 \le \tau \le t} |h(\tau)|, \quad \|h\|_t = |h|_t + \sup_{0 \le \tau_1, \tau_2 \le t} \frac{|h(\tau_1) - h(\tau_2)|}{|\tau_1 - \tau_2|^{(1+\alpha)/2}}.$$

Lemma 1. If the curves $x = h(t)$ and $x = g(t)$ are of class H_α, then the heat potential of a simple layer

$$F(x, t; h) = \int_0^t \frac{\varphi(\tau; h)}{\sqrt{t - \tau}}\, e^{-(x - h(\tau))^2 / 4(t - \tau)}\, d\tau$$

satisfies the inequality

$$|F(x, t; h) - F(x, t; g)| \le A |h - g|_t + B \max_{0 \le \tau \le t} |\varphi(\tau; h) - \varphi(\tau; g)|,$$

where the constants A and B depend on φ, M, K, T, and α.

Lemma 2. If the curves $x = h(t)$ and $x = g(t)$ belong to the class V, and the function $\varphi(\tau; t)$ satisfies a Hölder condition in τ with exponent $\delta > 1/2$, then the heat potential of a double layer

$$\Phi(x, t; h) = \int_0^t \frac{x - h(\tau)}{(t-\tau)^{3/2}} \, e^{-(x-h(\tau))^2/4(t-\tau)} \varphi(\tau; h) \, d\tau$$

satisfies the inequality

$$|\Phi(x, t; h) - \Phi(x, t; g)| \le C \max_{0 \le \tau \le t} |\varphi(\tau; h) - \varphi(\tau; g)| + o(\|h - g\|_t),$$

where C is a constant depending on φ, M, K, T, and δ.

Let in (1)–(4) $F(x)$, $\mu_i(t)$, $\lambda_i(t)$, $\psi_i(t)$, and $X_2(t)$ be fixed, while $X_1(t) \equiv h(t)$ varies; then the corresponding solution of problem (1)–(4) will be denoted by $u(x, t; h)$.

Theorem 1. Suppose the following conditions are satisfied:

1. $a(x, t)$ has derivatives $\partial a/\partial x$, $\partial a/\partial t$, and $\partial a/\partial x$, $\partial a/\partial t$, $b(x, t)$, $c(x, t)$, $f(x, t)$ are continuous and satisfy one of Gevrey' s conditions (A) (([1]), pp. 350, 351) (for example, a Hölder condition with nonzero exponent in one of the variables x or t); moreover, $0 < a_0 \le a(x, t) \le A_0$, where a_0, A_0 are constants.
2. The initial function $F(x)$ has derivatives $F'(x)$, $F''(x)$, and $F''(x)$ satisfies a Hölder condition in x with nonzero exponent.
3. The function $\mu_i(t)$ has a derivative $\mu_i'(t)$ satisfying a Hölder condition in t with nonzero exponent.
4. The functions $\psi_i(t)$, $\lambda_i(t)$, $X_i(t)$ $(i = 1, 2)$ satisfy a Hölder condition in t with exponent $> 1/2$.

Then, if the curves $x = h(t)$ and $x = g(t)$ belong to the class V, then

$$\lim_{\|h-g\|_t \to 0} \max_{S(t;h,g)} |u(\xi, \tau; h) - u(\xi, \tau; g)| = 0, \tag{5}$$

$$\lim_{\|h-g\|_t \to 0} \max_{S(t;h,g)} \left| \frac{\partial u(\xi, \tau; h)}{\partial x} - \frac{\partial u(\xi, t; g)}{\partial x} \right| = 0, \tag{6}$$

where

$$S(t; h, g) = \{(\xi, \tau), \ \max(h(\tau), g(\tau)) \le \xi \le X_2(\tau); \ 0 \le \tau \le t\}.$$

Remark. The existence of a solution $u(x, t; h)$ of problem (1)–(4), continuous together with the derivative $\partial u / \partial x$ up to the boundary of the domain, under the assumptions of Theorem 1, was proved by Gevrey (^{1}).

II. We now consider the solution $u_i(x, t)$ $(i = 1, 2)$ of the system (see (^{2})) of equations

$$\frac{\partial^2 u_i}{\partial x^2} = a_i(x,t)\frac{\partial u_i}{\partial x} + b_i(x,t)\frac{\partial u_i}{\partial x} + c_i(x,t)u + f_i(x,t), \tag{7}$$

$$X_j(t) < x < X_{j+1}(t), \qquad 0 < t < T,$$

satisfying the initial conditions

$$u_i(x,0) = F_i(x), \qquad X_j(0) \le x \le X_{j+1}(0), \tag{8}$$

the conjugation conditions

$$\alpha_1(t)\frac{\partial u_1(X_2(t),t)}{\partial x} - \alpha_2(t)\frac{\partial u_2(X_3(t),t)}{\partial x} = \beta(t), \tag{9}$$

$$u_1(X_2(t),t) - u_2(X_3(t),t) = \gamma(t), \qquad 0 \le t \le T, \tag{10}$$

and boundary conditions of one of the following types:

$$u_i(X_j(t),t) = \mu_i(t), \qquad 0 \le t \le T; \tag{11a}$$

$$\frac{\partial u_i(X_j(t),t)}{\partial x} + \lambda_i(t)u_i(X_j(t),t) = \psi_i(t), \qquad 0 \le t \le T, \tag{11b}$$

provided the compatibility conditions

$$\alpha_1(0)F_1'(X_2(0)) - \alpha_2(0)F_2'(X_3(0)) = \beta(0), \tag{12}$$

$$F_1(X_2(0)) - F_2(X_3(0)) = \gamma(0) \tag{13}$$

and, respectively,

$$F_i(X_j(0)) = \mu_i(0) \tag{14a}$$

or

$$F_i'(X_j(0)) + \lambda_i(0) F_i(X_j(0)) = \psi_i(0). \tag{14b}$$

(In formulas (7), (8), (11 a, b), (14 a, b), $j = 1$ when $i = 1$ and $j = 3$ when $i = 2$.) If the functions $F(x)$, $\mu_i(t)$, $\alpha_i(t)$, $\beta(t)$, $\gamma(t)$, $\psi_i(t)$, $\lambda_i(t)$ are fixed, then the solution $u_i(x,t)$ of problem (7)–(14) depends continuously (in the sense of (5), (6)) on any of the curves $x = X_i(t)$ $(i = 1,2,3,4)$ belonging to the class V. For definiteness, let us fix $X_i(t)$ $(i = 1,2,4)$ and denote the solution of problem (7)–(14) by $u_i(x,t;h)$, where $h(t) \equiv X_3(t)$.

Theorem 2. *Suppose that the following conditions are fulfilled:*

1. *For $a_i(x,t)$, $b_i(x,t)$, $c_i(x,t)$, $f_i(x,t)$, condition 1 of Theorem 1 is fulfilled, and in addition*

$$\sqrt{a_1(X_2(t),t)}\,\alpha_2(t) + \sqrt{a_2(X_3(t),t)}\,\alpha_1(t) \neq 0, \qquad 0 \le t \le T.$$

2. *For the initial function $F(x)$, condition 2 of Theorem 1 is fulfilled.*
3. *The functions $\gamma(t)$, $\mu_i(t)$ have derivatives $\gamma'(t)$, $\mu_i'(t)$ satisfying a Hölder condition in t with a nonzero exponent.*
4. *The functions $\alpha_i(t)$, $\lambda_i(t)$, $\beta(t)$, $\psi_i(t)$, $X_j(t)$ $(i = 1,2;\ j = 1,2,3,4)$ satisfy a Hölder condition in t with exponent $> 1/2$.*

Then, if the curves $x = h(t)$, $x = g(t)$ belong to the class V, then

$$\lim_{\|h-g\|_t \to 0} \max_{S^{(i)}(h,g;t)} |u_i(\xi,\tau;h) - u_i(\xi,\tau;g)| = 0,$$

$$\lim_{\|h-g\|_t \to 0} \max_{S^{(i)}(h,g;t)} \left| \frac{\partial u_i(\xi,\tau;h)}{\partial x} - \frac{\partial u_i(\xi,\tau;g)}{\partial x} \right| = 0, \qquad i = 1,2,$$

where

$$S^{(1)}(h,g;t) = \{(\xi,\tau),\ X_1(\tau) \le \xi \le X_2(\tau),\ 0 \le \tau \le t\},$$

$$S^{(2)}(h,g;t) = \{(\xi,\tau),\ \max(h(\tau),g(\tau)) \le \xi \le X_4(\tau),\ 0 \le \tau \le t\}.$$

Remark. The existence of a solution $u(x,t;h)$ of problem (7)–(14), continuous together with $\partial u_i/\partial x$ in the domain $\{X_j(t) \le x \le X_{j+1}(t);\ 0 \le t \le T\}$ $(j = 1$ when $i = 1;\ j = 3$ when $i = 2)$, under the hypotheses of Theorem 2, was proved in (^{2}). Theorems 1 and 2 are proved by a unified method using

using the theory of heat potentials developed by Gevrey for problem (1)–(4) in paper (^{1}) (see, respectively, (^{2}) in the case of problem (7)–(14)).

Let us briefly outline the proof of Theorem 1. With the aid of Gevrey' s substitution ([1], p. 370)

$$x = \int_0^y \sqrt{a(\xi, t)}\, d\xi$$

one may restrict oneself in equation (1) to the case $a(x, t) \equiv 1$. The solution $u(x, t; h)$ of problem (1)–(4) can then, as Gevrey ([1]) showed, be obtained by the method of successive approximations, setting

$$u^{(n)}(x, t; h) = \overline{u}^{(n)}(x, t; h) - v^{(n)}(x, t; h), \qquad n = 1, 2, \dots,$$

where

$$\overline{u}^{(n)}(x, t; h) = v(x, t; h) - \frac{1}{2\sqrt{\pi}} \iint_{S(t;h,h)} \left[b(\xi, \tau) \frac{\partial u^{(n-1)}(\xi, \tau; h)}{\partial \xi} + c(\xi, \tau) u^{(n-1)}(\xi, \tau; h) \right] \frac{e^{-(x-\xi)^2/4(t-\tau)}}{\sqrt{t-\tau}}\, d\xi\, d\tau,$$

$$v(x, t; h) \equiv u^{(0)}(x, t; h)$$

is the solution of the nonhomogeneous heat equation with the initial and boundary conditions (2), (3), while $v^{(n)}(x, t; h)$ is the solution of the homogeneous heat equation with zero initial data and nonhomogeneous boundary conditions (3), constructed with the aid of $\overline{u}^{(n)}(x, t; h)$ so that $u^{(n)}(x, t; h)$ satisfy the prescribed conditions (3). In this case $v^{(n)}(x, t; h)$ is representable as a sum of heat potentials of a simple layer. Application of Lemmas 1 and 2 shows the continuity of $v^{(n)}(x, t; h)$ with respect to h in the sense of (5), (6). It is then found that $\overline{u}^{(n)}(x, t; h)$, as well as $v(x, t; h)$, is also continuous with respect to h in the sense of (5), (6). Finally, the uniform convergence with respect to h of $u^{(n)}(x, t; h)$ and $\partial u^{(n)}(x, t; h)/\partial x$ to $u(x, t; h)$ and $\partial u(x, t; h)/\partial x$, respectively, makes it possible to complete the proof of Theorem 1.

The proof of Theorem 2 is carried out analogously, using the results of paper ([2]).

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