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Abstract

Full Text

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ON MULTIPOINT DIFFERENCE METHODS FOR SOLVING THE CAUCHY PROBLEM FOR THE EQUATION $y^{'} = f(x, y)$

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Mathematics

The Cauchy problem

$$y' = f(x, y), \qquad (x, y) \in G, \qquad G \text{ an open set}; \tag{1}$$

$$y(x_0) = y_0, \qquad (x_0, y_0) \in G, \tag{2}$$

will be solved approximately by replacing it with the difference Cauchy problem

$$\sum_{i=0}^{m} \alpha_i y_{k+i} = h \sum_{i=0}^{n} \beta_i f(x_{k+i}, y_{k+i}), \qquad x_j = x_0 + jh,\ h > 0; \tag{3}$$

$$y_i \cong y(x_i), \qquad i = 0, 1, \dots, q-1^*, \tag{4}$$

where m, n are prescribed integers, $m > 0,\ n \geqslant 0,\ n \gtreqless m$; α_i, β_i are prescribed real numbers; $\alpha_0 \cdot \alpha_m, \beta_n$ are different from zero; $q = \max(m, n)$ is the order of the difference equation (3); $y = y(x)$ is the exact solution of problem (1), (2); y_i is the exact solution of problem (3), (4).

In the case when $n > m$, the constants multiplying the powers of h in the approximation error are smaller than in the cases $n \leqslant m$ [2,3].

We shall say that $f(x, y)$ belongs to the class (A') if $f(x, y)$ is continuous in the domain G with respect to the aggregate of variables (x, y) and satisfies the Lipschitz condition in y [1].

Let us introduce, for the error of the method (3), (4), the notation

$$\delta_i = y(x_i) - y_i.$$

Definition. We shall say that the difference method (3), (4) converges unconditionally in the class (A') if, whatever the manner in which $\max_{0 \leqslant i \leqslant q-1} |\delta_i|$ tends to zero, for every function $f(x, y) \in (A')$ there exists an interval $x_0 \leqslant x \leqslant \bar{x}_f$ such that $\delta_i \to 0$ as $h \to 0$, $q \leqslant i \leqslant T_h$, where

$$T_h = \left[\frac{\bar{x}_f - x_0}{h}\right] - \frac{1}{2}\,\mathrm{sign}(n-m)\,[1 + \mathrm{sign}(n-m)].$$

Theorem 1. *In order that the difference method (3), (4) converge unconditionally in the class (A'), it is necessary and sufficient that:*

1) the relations

$$\sum_{i=0}^{m} \alpha_i = 0, \qquad \sum_{i=0}^{m} i\alpha_i = \sum_{i=0}^{n} \beta_i \neq 0; \tag{5}$$

be satisfied;

2) all roots of the characteristic equation

$$\sum_{i=0}^{m-1} \sum_{j=i+1}^{m} \alpha_j \lambda^i = 0 \tag{6}$$

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All the propositions given below are also valid for another way of choosing the initial conditions $y_i \cong y(x_i)$, $i = 0, -1, \dots, -(q-1)$ (see, for example, ([5,9])).

modulus did not exceed unity, and the multiple ones among them were, in modulus, strictly less than unity.

Let, in solving problem (3), (4), the computation be carried out with round-off, and let the approximate values y_i obtained in this way be denoted by y_i^*. We introduce for the total error the notation

$$D_k = y(x_k) - y_k^*.$$

Theorem 2. *If the difference method (3), (4) converges unconditionally in the class (A') and if, in computing with round-off in the process of solving problem (3), (4), the condition*

$$|\gamma|_k \leqslant o(h) \quad \text{as } h \to 0, \tag{7}$$

is satisfied, where

$$\gamma_k = \sum_{i=0}^{m} \alpha_i y^*_{k+i} - h \sum_{i=0}^{n} \beta_i f(x_{k+i}, y^*_{k+i}),$$

then

$$\lim_{h\to 0} \max_{q\leqslant k\leqslant T_h} |D_k| = 0$$

for any way in which

$$\max_{0\leqslant k\leqslant q-1} |D_k|$$

tends to zero. Thus, condition (7) ensures the stability of the computational process.

We shall say that $f(x,y) \in (A'')$ if $f(x,y)$ satisfies the Hölder condition in the domain G with respect to both arguments.

For δ_k and D_k, when $f(x,y) \in (A'')$, estimates have been obtained, which we do not write out because of their cumbersomeness. Using the estimate for D_k, it is not difficult to establish that the following is true.

Theorem 3. *If all roots of equation (6) are, in modulus, less than unity, and $f(x,y) \in (A'')$, then, under conditions (5) and (7),*

$$\frac{\Delta D_k}{h} \to 0 \quad \text{as } h \to 0 \tag{8}$$

outside an arbitrarily small neighborhood of $x = x_0$, whatever the way in which

$$\max_{0\leqslant k\leqslant q-1} |D_k|$$

tends to zero. If, in addition to the preceding conditions,

$$\Delta D_k / h \to 0 \quad \text{as } h \to 0, \quad k = 0, 1, \dots, q-2,$$

then relation (8) holds for $0 \leqslant k \leqslant T_h$.

We shall say that $f(x,y) \in (B_l)$ if it satisfies in the domain G the Lipschitz condition with respect to both arguments. Denote the Lipschitz constant by L.

Put

$$\rho_k = \sum_{i=0}^{n} \alpha_i y(x_{k+i}) - h \sum_{i=0}^{n} \beta_i f(x_{k+i}, y_{k-i}) \quad \text{and} \quad R_k = \rho_k - \gamma_k.$$

Define the functions $g_j(k)$ and $g(k)$, $j = 0, 1, \dots, m-1$, as follows: 1) $g_j(k)$, $j = 0, 1, \dots, m-1$, is the solution, as a function of k, of the equation

$$\sum_{i=0}^{m} \alpha_i z_{k+i} = 0,$$

satisfying, respectively, the initial conditions:

$$g_j(\nu) = \delta_{j\nu},$$

where $\delta_{j\nu}$ is the Kronecker symbol, $\nu = 0, 1, \dots, m-1$; 2) $g(k)$ is the solution of the equation

$$\sum_{i=0}^{m} \alpha_i z_{k+i} = \delta_{0k},$$

satisfying the initial conditions

$$g(\nu) = 0, \quad \nu = 0, 1, \dots, m-1.$$

Under unconditional convergence of the difference method (3), (4) in the class (A'),

$$|g_j(k)| \leqslant g_j < +\infty, \quad j = 0, 1, \dots, m-1;$$

$$|g(k)| \leqslant g < +\infty, \quad k = 0, 1, \dots$$

Theorem 4. *If $|D_k| \leqslant \overline{\varphi}(h)$ for $k = 0, 1, \dots, q-1$ and $|R_k| \leqslant h\overline{\overline{\varphi}}(h)$ for $0 \leqslant k \leqslant T_h$, and $f(x, y) \in (B_l)$, then the following estimates hold:*

$$|D_{k+m}| \leqslant \left\{ \begin{array}{c} \overline{P}_k \\ \overline{\overline{P}}_k \end{array} \right\} C_{m,n} e^{C_{m,n} gBL(x_k - x_0)} \quad \text{for } n \leqslant m,$$

where

$$C_{m,n} = (1 - h|\beta_m|L)^{-[1+\operatorname{sign}(n-m)]},$$

$$|D_{k+m}| \leqslant \left\{ \begin{array}{c} \overline{P}_k \\ \overline{\overline{P}}_k \end{array} \right\} \left[e^{gBL(x_k - x_0)} + \dots + (hgBL)^{s-1} e^{sgBL(x_k - x_0)} \right] +$$

$$+(hgBL)^s\, 2b\,(n-m+1)e^{sgBL(x_k-x_0)} \quad \text{for } n>m;$$

$s \geqslant 1^*$ is an arbitrary natural number, $|y_k^* - y_0| \leqslant b$,

$$\overline{P}_k = \left\{ hgmBL + \max_{0\leqslant k'\leqslant k} \sum_{j=1}^{m} |g_j(k'+m)| \right\} \overline{\varphi}(h) + (x_k - x_0) g\overline{\overline{\varphi}}(h),$$

$$\overline{\overline{P}}_k = \left\{ hgmBL + \sum_{j=1}^{m} g_j \right\} \overline{\varphi}(h) + (x_k - x_0) g\overline{\overline{\varphi}}(h), \qquad B = \sum_{i=0}^{n} |\beta_i|,$$

and the brace $\left\{ \begin{matrix} \overline{P}_k \\ \overline{\overline{P}}_k \end{matrix} \right\}$ means that either $\overline{P}_k$ or $\overline{\overline{P}}_k$ may be taken as the factor; in the case of $\overline{\overline{P}}_k$ the estimates obtained are somewhat coarser.

We shall say that: 1) $f(x,y) \in (B_{s-1})$, if the partial derivatives of f with respect to x and y up to order $(s-1)$, inclusive, satisfy the Lipschitz condition; 2) $y(x) \in (b_s)$, if $y(x)$ on some interval $x_0 \leqslant x \leqslant \overline{x}$ has an s-th derivative satisfying the Lipschitz condition. If $f(x,y) \in (B_{s-1})$, then the solution of problem (1), (2), $y(x) \in (b_s)$. We shall say that: 1) $y(x) \in (b_{s,M})$, if $y(x) \in (b_s)$ and the Lipschitz constant of the s-th derivative $y(x)$ is equal to M; 2) $f(x,y) \in (B_{s-1,M})$, if $f(x,y) \in (B_{s-1})$ and the solution of problem (1), (2), $y(x) \in (b_{s,M})$.

Introduce the operator

$$L_h(y(x)) \equiv \sum_{i=0}^{m} \alpha_i y(x_{k+i}) - h \sum_{i=0}^{n} \beta_i y'(x_{k+i}),$$

defined on differentiable functions $y(x)$.

Theorem 6. If $y(x) \in (b_{s,M})$ and equation (3) has degree s (7), then the estimate

$$|L_h(y(x))| \leqslant h^{s+1} C_k(h,M), \tag{9}$$

holds, where $C_k(h,M)$, besides k, h, M, contains the coefficients α_i, β_i of equation (3), and $C_k(h,M) = O(1)$ as $h \to 0$, and there exists $y(x) \in (b_{s,M})$ for which (9) turns into an equality.

Corollary of Theorems 5 and 6. If $f(x,y) \in (B_{s-1})$, and equation (3) has degree $\geqslant s$, then under the conditions $\overline{\varphi}(h) \leqslant C^* h^s$, $|\gamma_k| \leqslant C^{**} h^{s+1}$, $s \geqslant 1$, for D_k one obtains the uniform estimate

$$|D_k| \leqslant O(h^s).$$

Theorem 7. If the difference equation (3) has the form

$$y_{k+m} = \sum_{i=0}^{m-1} \alpha_i y_{k+i} + h \sum_{i=0}^{m} \beta_i f(x_{k+i}, y_{k+i}),$$

where

$$f_y(x, y) \leqslant 0, \quad \alpha_i > 0;\ i = 0, 1, \dots, m-1; \quad \beta_i \geqslant 0,\ i = 0, 1, \dots, m,$$

then

$$\max_{k \leqslant \nu \leqslant k+m-1} |D_\nu| \leqslant \max_{0 < \nu \leqslant m-1} |D_\nu| + k \sup_k |R_k|, \qquad k = 0, 1, \dots$$

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* It is advisable to take s equal to the degree of equation (3); see Theorem 6 in this connection.

Examples

$$y_{k+1} = y_k + hf(x_k, y_k),$$

$$y_{k+1} = y_k + hf(x_{k+1}, y_{k+1}),$$

$$y_{k+1} = y_k + h\left(\frac{1}{2} f(x_{k+1}, y_{k+1}) + \frac{1}{2} f(x_k, y_k)\right),$$

$$y_{k+2} = \frac{1}{2} y_{k+1} + \frac{1}{2} y_k + h\left(\frac{3}{8} f_{k+2} + f_{k+1} + \frac{1}{8} f_k\right).$$

Let us dwell briefly on a posteriori estimates. Put $\bar{\Theta}(x) = \bar{y}'(x) - f(x, \bar{y}(x))$, where $y = \bar{y}(x)$ is the equation of the broken line joining the points (x_i, y_i^*), $i = 0, 1, \dots$. Put $D(x) = y(x) - \bar{y}(x)$,

$$F^{(D)}(x) = [f(x, y(x)) - f(x, \bar{y}(x))]\,[y(x) - \bar{y}(x)]^{-1},$$

$$\bar{\Theta}^+(x) = \begin{cases} \bar{\Theta}(x), & \text{if } \bar{\Theta}(x) \geq 0, \\ 0, & \text{if } \bar{\Theta}(x) < 0, \end{cases} \qquad \bar{\Theta}^-(x) = \begin{cases} 0, & \text{if } \bar{\Theta}(x) \geq 0, \\ \bar{\Theta}(x), & \text{if } \bar{\Theta}(x) < 0. \end{cases}$$

Under the formulated conditions the following is valid.

Theorem 8. For $D(x)$ the estimate holds

$$|D(x)| \leq |D(x_0)| \exp\left[\int_{x_0}^{x} F^{(D)}(\xi)\,d\xi\right] + \left|\int_{x_0}^{x} \bar{\Theta}(\eta) \exp\left[\int_{\eta}^{x} F^{(D)}(\xi)\,d\xi\right]\,d\eta\right|. \tag{10}$$

Put

$$\overline{K}_0(\xi) = \max f_y(\xi, y), \qquad \underline{K}_0(\xi) = \min f_y(\xi, y) \quad \text{in } G.$$

Next define recursively

$$\overline{K}_n(\xi) = \max f_y(\xi, y), \qquad \underline{K}_n(\xi) = \min f_y(\xi, y),$$

$$\bar{y}(\xi) - \sigma_{n-1}(\xi) \leq y \leq \bar{y}(\xi) + \sigma_{n-1}(\xi),$$

$$\sigma_n(x) = |D(x_0)| \exp\left[\int_{x_0}^{x} \overline{K}_n(\xi)\,d\xi\right] +$$

$$+ \max\left\{\int_{x_0}^{x} \bar{\Theta}^{+}(\eta) \exp\left[\int_{\eta}^{x} \overline{K}_n(\xi)\,d\xi\right]\,d\eta,\ \int_{x_0}^{x} |\bar{\Theta}^{-}(\eta)| \exp\left[\int_{\eta}^{x} \underline{K}_n(\xi)\,d\xi\right]\,d\eta\right\}$$

$$- \min\left\{\int_{x_0}^{x} \bar{\Theta}^{+}(\eta) \exp\left[\int_{x_0}^{x} \underline{K}_n(\xi)\,d\xi\right]\,d\eta,\ \int_{x_0}^{x} |\bar{\Theta}^{-}(\eta)| \exp\left[\int_{x_0}^{x} \underline{K}_n(\xi)\,d\xi\right]\,d\eta\right\}.$$

Theorem 9. For the functions $\sigma_i(x)$ defined above, the relations hold

$$\sigma_0(x) \geq \sigma_1(x) \geq \cdots \geq \sigma_n(x) \geq \cdots, \qquad |D(x)| \leq \sigma_n(x), \qquad n = 1, 2 \ldots$$

By the method set forth in Theorems 8 and 9, other a posteriori estimates are also obtained.

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Note: Figure translations are in progress. See original paper for figures.