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Abstract

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ON THE EXACTNESS OF LIMIT SEQUENCES OF THE INVERSE SPECTRUM OF EXACT SEQUENCES OF DISCRETE GROUPS

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In the present note only discrete and bicompact Abelian groups are considered. In considering spectra of sequences of bicompact groups, the homomorphisms acting in the spectra of groups and in the sequences of groups are assumed to be continuous. The question of the exactness of limit sequences of spectra of exact sequences of groups is of great importance in homology theory. In this connection a number of important results have been obtained. We give formulations of the main of these results, the proof of which can be found, for example, in ([3]), Ch. 8.

Let a direct spectrum $\{|X_k, \rightarrow|_\alpha, \Pi^\alpha_\beta\}$ of exact sequences of groups

$$|X_k, \rightarrow|_\alpha = \cdots \rightarrow X_{k,\alpha} \rightarrow X_{(k+1),\alpha} \rightarrow X_{(k+2),\alpha} \rightarrow \cdots,$$

taken over the directed set of indices $A = \{\alpha, >\}$, be given; then the limit sequence, which we denote by

$$[X_k, \rightarrow] = \varinjlim\{|X_k, \rightarrow|_\alpha, \Pi^\alpha_\beta\} \quad (\beta > \alpha)$$

of this spectrum is an exact sequence.

Let an inverse spectrum $\{|\overline{X}_k, \leftarrow|_\alpha, \Pi^\beta_\alpha\}$ $(\beta > \alpha)$, $\alpha,\ \beta \in A$, of exact sequences of groups

$$|\overline{X}_k, \leftarrow|_\alpha = \cdots \leftarrow \overline{X}_{k,\alpha} \leftarrow \overline{X}_{(k+1),\alpha} \leftarrow \overline{X}_{(k+2),\alpha} \leftarrow \cdots$$

be given; then the limit sequence

$$[\overline{X}_k, \leftarrow] = \varprojlim\{|\overline{X}_k, \leftarrow|_\alpha, \Pi^\beta_\alpha\}$$

is a semi-exact sequence. In the case where an inverse spectrum of exact sequences of bicompact groups is considered, the limit sequence of this spectrum will be an exact sequence.

In these results no restrictions are imposed either on the homomorphisms acting in the spectra of groups or on the sets of indices over which the spectra are taken. The limit sequence of the inverse spectrum of exact sequences of discrete groups in this case is a semi-exact sequence, and this result, as numerous examples show, cannot be strengthened without additional substantial restrictions on inverse spectra of exact sequences of groups.

In the present note a condition is found on an inverse spectrum of exact sequences of discrete groups, taken over a directed set of indices $A' = \{\nu, >\}$, having a countable cofinal part, under the fulfillment of which the limit sequence of this spectrum will be an exact sequence of groups. Then the result is applied to the derivation of a duality relation of the type of theorems of position for compact sets lying in a finite polyhedron. Such a relation has not previously been indicated in the literature.

Let an inverse spectrum be given

$$\{|Y_k, \leftarrow|_\nu, \Pi_\nu^{\nu'}\}\ (\nu' > \nu), \quad k > m, m+1, \dots, n \text{ —integers}, \tag{1}$$

exact sequences of groups

$$[Y_{h,\leftarrow}]_\alpha = Y_{m,\alpha} \leftarrow Y_{(m+1)\alpha} \leftarrow \cdots \leftarrow Y_{k,\alpha} \leftarrow \cdots \leftarrow Y_{n,\alpha}, \tag{2}$$

where the set of indices $A' = \{\nu, >\}$ is a directed set having a countable cofinal part. We shall denote the limiting sequence of the spectrum (1) as follows:

$$[Y_{k,\leftarrow}] = \lim_{\leftarrow}\{[Y_{k,\leftarrow}]_\nu, \Pi_\nu^{\nu'}\} \quad (\nu' > \nu). \tag{3}$$

Let us agree on the following notation:

$$B^* = \mathrm{Ker}(B \to C), \qquad B_* \to \mathrm{Im}(A \to B),$$

where A, B, C are groups; $\mathrm{Ker}(B \to C)$ is the kernel of the homomorphism $B \to C$; $\mathrm{Im}(A \to B)$ is the image of the homomorphism $A \to B$.

Exactness of the sequence (2) means that for every k one has $Y_k^* = Y_{*k}$.

With respect to the limiting sequence (3), one can assert only that it is semi-exact, i.e. the inclusion $Y_{*k} \subset Y_k^*$ holds. There are examples showing that in our case as well this result cannot be strengthened.

Here we shall indicate a condition on the homomorphisms acting in inverse spectra of groups under which the limiting sequence of the inverse spectrum of exact sequences of discrete groups will be an exact sequence.

Lemma 1. Suppose it is given that in the $(k+2)$-nd spectrum $\{Y_{k+2}^*, \Pi_{\nu,k+2}^{*\nu'}\}$ $(\nu > \nu')$ the homomorphisms

$$\Pi_{(k+2),\nu}^{*\nu'} : Y_{(k+2),\nu'}^* \to Y_{(k+2),\nu}^* \quad (\nu' > \nu),$$

generated by the homomorphisms

$$\Pi^{\nu'}_{(k+2),\nu} : Y_{(k+2),\nu'} \to Y_{(k+2),\nu}$$

are epimorphisms. Then the limiting sequence

$$Y_k \leftarrow Y_{k+1} \leftarrow Y_{k+2} \tag{4}$$

is an exact sequence.

Indeed, in view of what was said above, the sequence (4) is a semi-exact sequence; therefore, to prove Lemma 1 it remains only to prove that the inclusion $Y^*_{(k+1)} \subset Y_{*(k+1)}$ holds. For this it is necessary to construct, for each element $y^*_{(k+1)}$ of the subgroup $Y^*_{(k+1)}$ with coordinates $\{y^*_{(k+1),\nu}\}$, an element

$$y_{(k+2)} = \{y_{(k+2),\nu}\}$$

of the group Y_{k+2} which, under the homomorphic mapping $Y_{k+2} \to Y_{k+1}$, goes into the given element y^*_{k+1} of the subgroup Y^*_{k+1}.

Let $y^*_{(k+1),\nu}$ and $y^*_{(k+1),\nu'}$ be, respectively, the ν-th and ν'-th coordinates of the given thread

$$y^*_{k+1} = \{y^*_{(k+1),\nu}\};$$

let $y_{(k+2),\nu}$ be an element of the group $Y_{(k+2),\nu}$ which, under the homomorphic mapping

$$Y_{(k+2),\nu} \to Y_{(k+1),\nu},$$

goes into the element $y^*_{(k+1),\nu}$, which is the ν-th coordinate of the given thread

$$y^*_{k+1} = \{y^*_{(k+1),\nu}\}.$$

Such an element exists by virtue of the exactness of the sequence (2). Denote by $Y'_{(k+2),\nu'}$ the full preimage of the element $y^*_{(k+1),\nu'}$ in the group $Y_{(k+2),\nu'}$ under the homomorphic mapping

$$Y_{(k+2),\nu'} \to Y_{(k+1),\nu'}$$

(this set is nonempty by virtue of the exactness of the sequence (2)). Consider some element $y'_{(k+2),\nu'}$ of the set $Y'_{(k+2),\nu'}$. The element

$$y_{(k+2),\nu} - \Pi^{\nu'}_{(k+2),\nu} y'_{(k+2),\nu'}$$

is an element of the subgroup $Y^*_{(k+2),\nu}$; therefore, by the condition of the lemma, in the subgroup $Y^*_{(k+2),\nu'}$ there exists an element $y^*_{(k+1),\nu'}$ which, under the homomorphic mapping

$$Y^*_{(k+2),\nu'} \to Y^*_{(k+2)\nu},$$

goes into the element

$$y_{(k+2),\nu} - \Pi^{\nu'}_{(k+2),\nu} y'_{(k+2),\nu'}.$$

As is not difficult to see, the element

$$y_{(k+2),\nu'} = y'_{(k+2),\nu'} + y^*_{(k+2),\nu'}$$

under the homomorphic mapping

$$Y_{(k+2),\nu'} \to Y_{(k+2),\nu}$$

goes into the given element $y_{(k+2),\nu}$. In the same way one constructs an element $y_{(k+2),\nu''}$ of the group $Y_{(k+2),\nu''}$ for $\nu'' > \nu'$, and so on. Thus we construct elements in all groups corresponding to the linearly ordered—

ordered set of indices, which is a cofinal part in the set of indices A'. Then, taking projections of the constructed elements into all groups with smaller indices, for which we did not construct elements, we obtain the desired thread. Thus Lemma 1 is proved.

We shall now formulate and prove the following lemma:

Lemma 2. Suppose it is given that in the $(k+3)$-rd spectrum $\{Y_{(k+3),\nu}, \Pi^{\nu'}_{(k+3),\nu}\}$ $(\nu' > \nu)$ the homomorphisms $\Pi^{\nu'}_{(k+3),\nu} : Y_{(k+3),\nu'} \to Y_{(k+3),\nu}$ $(\nu' > \nu)$ are epimorphisms; then the limiting sequence

$$Y_k \leftarrow Y_{k+1} \leftarrow Y_{k+2} \tag{5}$$

is an exact sequence.

Proof. From the condition of Lemma 2 it is easy to derive that in the $(k+2)$-nd spectrum $\{Y^*_{(k+2),\nu}, \Pi^{*\nu'}_{(k+2),\nu}\}$ the homomorphisms $\Pi^{*\nu'}_{(k+2),\nu} : Y^*_{(k+2),\nu'} \to Y^*_{(k+2),\nu}$ are epimorphisms; therefore, by Lemma 1, sequence (5) will be an exact sequence, as was required to prove.

We combine the results obtained in Lemmas 1 and 2 in the following theorem:

Theorem. Suppose it is given that in the inverse spectrum (1) the homomorphisms

$$\Pi^{\nu'}_{n,\nu} : Y^*_{n,\nu'} \to Y^*_{n,\nu} \quad \text{and} \quad \Pi^{\nu'}_{i,\nu} : Y_{i,\nu'} \to X_{i,\nu}$$

for all $i = m+3, \dots, n$ are epimorphisms. Then the limiting sequence (3) is an exact sequence.

Let us derive some corollaries from the theorem obtained.

Corollary 1. Suppose it is given that in the inverse spectrum of exact sequences of the form $0 \to A_\nu \to B_\nu \to C_\nu \to 0$ the homomorphisms $\Pi^{\nu'}_\nu : A_{\nu'} \to A_\nu$ in the spectrum $\{A_\nu, \Pi^{\nu'}_\nu\}$ are epimorphisms. Then the limiting sequence $0 \to A \to B \to C \to 0$ of this spectrum is an exact sequence.

Corollary 2. Suppose an inverse spectrum of exact sequences of the form $0 \to X_\nu \to Y_\nu \to 0$ is given. Then the limiting sequence $0 \to X \to Y \to 0$ of this spectrum is exact, i.e. the limiting groups X and Y are isomorphic.

2. We shall give some applications of the results obtained to homology theory, in particular to the proof of the theorem on the location of a compactum in a finite polyhedron.

Let K be a finite n-dimensional polyhedron; F a compactum lying in the polyhedron K; $G = K \setminus F$ an open set complementary to F in the polyhedron K. Consider the directed set $\{K_\nu, >\}$ of triangulations of the polyhedron K. Denote by F_ν the closed subcomplex of the triangulation K_ν consisting of all simplices having nonempty intersection with the compactum F; by $G_\nu = K_\nu \setminus F_\nu$ denote the complementary open subcomplex of the triangulation K_ν. On each triangulation K_ν there is an exact sequence of homology groups

$$\cdots \leftarrow H_{r-1}(K_\nu) \leftarrow H_{r-1}(F_\nu) \leftarrow H_r(G_\nu) \leftarrow H_r(K_\nu) \leftarrow \cdots. \tag{6}$$

The homology groups are taken with an arbitrary coefficient group, and the 0-dimensional homology group is assumed reduced. The directed set of exact sequences (6), together with the homomorphisms $e_\nu^{\nu'}$, generated by the identity map of the body of the complex $K_{\nu'}$ into the body of the complex K_ν, forms an inverse spectrum of exact sequences

$$\{|H, \leftarrow|_\nu, e_\nu^{\nu'}\}. \tag{7}$$

In the spectra $\{H_r(K_\nu) e_\nu^{\nu'}\}$, for all $r = 0, 1, \dots, n$, the homomorphisms $e_\nu^{\nu'}$: $H_r(K_{\nu'}) \to H_r(K_\nu)$ are isomorphisms; therefore we are in the conditions of Lemma 2, and hence the following sequence is exact:

$$H_{r-1}(K) \leftarrow H_{r-1}(F) \leftarrow H_r(G), \qquad r = 1, \dots, n. \tag{8}$$

Hence we obtain that the groups $\operatorname{Ker}(H_{r-1}(F) \to H_{r-1}(K))$ and $H_r(G)/\operatorname{Ker}(H_r(G) \to H_{r-1}(F))$ are isomorphic.

This result is a transfer to a compact set F, lying in the polyhedron K, of the well-known relation (see (1)) for a subpolyhedron F lying in a polyhedron K, and therefore may be called the general duality law.

Now let the polyhedron K be an n-dimensional homological sphere, i.e., a polyhedron whose homology groups in all dimensions from 0 to $n-1$ are trivial (the 0-dimensional homology group is assumed to be reduced). Then, for the inverse spectrum (7), all the conditions of Corollary 2 are satisfied; therefore the limiting sequence

$$0 \leftarrow H_{r-1}(F) \leftarrow H_r(G) \leftarrow 0, \qquad r = 1, \dots, n-1$$

is exact. Hence we obtain that the groups $H_{r-1}(F)$ and $H_r(G)$ are isomorphic. In this case the general duality law coincides with Sitnikov' s second duality law in Δ-form for the case of a compact set F lying in the n-dimensional sphere

K. Here the homology groups are taken with respect to an arbitrary discrete or bicompact domain of coefficients. In the case when the homology groups are considered with respect to a bicompact domain of coefficients, Pontryagin' s duality law is easily derived from this.

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Note: Figure translations are in progress. See original paper for figures.