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**Abstract**

**Full Text**

**MECHANICS**

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# ON A NEW ANALOGY IN MECHANICS

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It is possible to establish a certain analogy between the plane problem of the potential flow of an ideal and incompressible fluid in hydromechanics and the plane problem of trajectories (for conservative fields) in classical mechanics.

Let us consider the motion of a material point of mass $m$ in a conservative field with potential function $U = U(x, y)$, for a given value of the energy $E$.

We shall write the differential equation of the trajectories, starting from the variational principle of least action in Jacobi' s form ([1]):

$$\delta \int_{(A)}^{(B)} \sqrt{2m(E-U)}\, ds = 0. \tag{1}$$

Composing the Euler equation for the variational problem (1) under consideration and introducing the function ([2])

$$\Phi(x, y) = \ln \sqrt{2(E - U(x, y))}, \tag{2}$$

equal to the natural logarithm of the magnitude of the velocity $v$ of the point (the mass $m$ is taken equal to unity), i.e. $\Phi = \ln v$, we obtain the differential equation of the trajectories in the following form ([2]):

$$y'' = (1 + {y'}^2)\left(-y' \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\right) \tag{3}$$

(the prime denotes differentiation with respect to $x$).

If we introduce the angle $\Psi$, formed by the velocity vector $\mathbf{v}$ with the $x$-axis, i.e. $\Psi = \operatorname{arc\,tg} y'$, then equation (3) takes the form

$$d\Psi = \frac{\partial \Phi}{\partial y}\, dx - \frac{\partial \Phi}{\partial x}\, dy. \tag{4}$$

Let us write the total differential of $\Phi$:

$$d\Phi = \frac{\partial \Phi}{\partial x}\, dx + \frac{\partial \Phi}{\partial y}\, dy \tag{5}$$

and, with the aid of (4), form the complex function

$$d\Phi - i\, d\Psi = \left( \frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y} \right) dz, \tag{6}$$

where $z = x + iy$ is the complex variable.

We shall consider those force fields for which the expression $\frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y}$ is an analytic function $\Omega(z)$ of the complex variable $z$, i.e.

$$\frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y} = \Omega(z). \tag{7}$$

This, obviously, will hold for force fields when $\Phi(x, y)$ is a harmonic function and $\Delta\Phi = 0$.

From the expression for the Laplacian of $U$,

$$\Delta U = -e^{2\Phi} \left\{ \Delta\Phi + 2 \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 \right] \right\},$$

it follows that if $\Phi(x, y)$ is a harmonic function, then the potential function $U$ will not be a harmonic function; and, conversely, if the potential function $U$ is harmonic, $\Delta U = 0$, then $\Phi(x, y)$ will not be harmonic ($\Delta\Phi < 0$).

From (6), when condition (7) is satisfied, it follows that the function

$$\Phi - i\Psi = \ln \left( v e^{-i\Psi} \right), \tag{8}$$

equal to the natural logarithm of the complex velocity of the point, will be an analytic function of the complex variable $z$:

$$\Phi - i\Psi = \int \Omega(z)\, dz = W(z). \tag{9}$$

On the other hand, the analytic function $W(z)$ may be regarded as the complex potential of a certain potential flow of an ideal and incompressible fluid ([3]), i.e.,

$$W(z) = \varphi(x, y) + i\chi(x, y), \tag{10}$$

where $\varphi(x, y)$ is the velocity potential and $\chi(x, y)$ is the stream function.

Thus, if we set

$$\Phi(x, y) = \varphi(x, y), \qquad \Psi(x, y) = -\chi(x, y), \tag{11}$$

then we thereby obtain the analogy indicated above. Namely, from a plane problem of hydromechanics one can pass to the plane problem of trajectories in classical mechanics if the family of lines of equal velocity potential $\varphi(x, y) =$ const is regarded as a family of equipotential lines $U(x, y) =$ const (since the condition $\Phi(x, y) =$ const, on the basis of the energy integral, is equivalent to the condition $U(x, y) =$ const), and the streamlines $\chi(x, y) =$ const are regarded as the isolines $\Psi(x, y) =$ const of the trajectories.

The converse assertion also holds: from the plane problem of classical mechanics concerning trajectories one can pass (in the sense indicated above) to a plane problem of hydromechanics, with the sole restriction that the function $\Phi(x, y) = \ln v$ be harmonic.

Thus, for example, if we consider the complex potential of a source of strength $Q$ placed at the origin $z = 0$,

$$W(z) = \frac{Q}{2\pi} \ln z \quad (z = re^{i\alpha}),$$

then the corresponding problem of classical mechanics is obtained if we take

$$\Phi = \frac{Q}{2\pi} \ln r + C_1, \qquad \Psi = -\frac{Q}{2\pi}\alpha + C_2$$

($C_1$ and $C_2$ are constants), which will correspond to the case of the motion of a point in a central field with potential $U = Cr^n$ ($C < 0,\ n = Q/\pi$). In particular, for a Coulomb field $U = C/r$, the analogue in hydromechanics will be a sink of strength $Q = -\pi$ ($n = -1$).

If one further makes use of the optical-mechanical analogy, then one can thereby establish a connection between plane problems of geometrical optics and plane problems of hydromechanics, for which, in formulas (11)

one should set $\Phi(x, y) = \ln n(x, y)$, where $n(x, y)$ is the refractive index of the optical medium.

Similarly, one can establish a connection with problems of electron optics as well, if one notes that an electric field with potential $\varphi_1$ corresponds to an optical medium with refractive index $n = c\sqrt{\varphi_1}$ ([4]).

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*