MATHEMATICS

G. S. LITVINCHUK

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Abstract

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G. S. LITVINCHUK

ON A PROBLEM GENERALIZING THE CARLEMAN BOUNDARY-VALUE PROBLEM

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§ 1. Let L be a simple closed Lyapunov curve dividing the plane into two domains: the interior D^+, containing the origin of coordinates, and the exterior D^-, containing the point at infinity. Consider the following boundary-value problem.

Find a piecewise analytic function $\Phi(z)$ from the boundary condition

$$a(t)\Phi^+(t) + b(t)\Phi^+[\alpha(t)] + c(t)\Phi^-(t) + d(t)\Phi^-[\alpha(t)] = h(t) \quad \text{on } L, \tag{1}$$

where the function $\alpha(t)$ maps the contour L one-to-one onto itself with reversal of the direction of traversal on it and has a derivative $\alpha'(t)$, different from zero at the points of L and satisfying on L a Hölder condition ($\alpha'(t) \subset H$). It is also assumed that on L the condition

$$\alpha[\alpha(t)] = t \tag{2}$$

is fulfilled.

The coefficients $a(t), b(t), c(t), d(t) \subset H$, do not vanish at the points of L and satisfy on L the conditions

$$a(t)a[\alpha(t)] = b(t)b[\alpha(t)], \qquad c(t)c[\alpha(t)] = d(t)d[\alpha(t)]; \tag{3}$$

the function $h(t) \subset H$.

We note that the transformation $\alpha(t)$ necessarily has on L two fixed points t_1 and t_2.

Problem (1) may be regarded as a generalization of the well-known boundary-value problem of determining a function $\Phi(z)$, analytic inside the domain D^+, from the boundary condition on the contour L

$$\Phi^+[\alpha(t)] = G(t)\Phi^+(t) + g(t), \tag{4}$$

where, by virtue of assumption (2), on L it must be that

$$G(t)G[\alpha(t)] = 1, \qquad g(t) + G(t)g[\alpha(t)] = 0. \tag{5}$$

The boundary-value problem (4) for the bounded domain D^+, under condition (2), was posed by Carleman ([1])* and solved in the works of D. A. Kveselava ([2,3]). The method developed by D. A. Kveselava can also be applied to the study of the Carleman problem for the unbounded domain D^-; the features distinguishing the exterior Carleman problem from the interior one are approximately the same as those distinguishing the exterior Dirichlet problem from the interior one ([4]).

The boundary-value problem (1), under conditions (2) and (3), reduces to an equivalent pair of Carleman boundary-value problems for finding the functions $\Phi^+(z)$ and $\Phi^-(z)$.

* Carleman considered only the homogeneous ($g(t) \equiv 0$) problem (4).

analytic respectively in the domains D^+ and D^-;

$$\Phi^+[\alpha(t)] = G_+(t)\Phi^+(t) + g_+(t) \quad \text{on } L; \tag{6}$$

$$\Phi^-[\alpha(t)] = G_-(t)\Phi^-(t) + g_-(t) \quad \text{on } L, \tag{7}$$

where

$$G_+(t) = -\frac{a(t)}{b(t)}, \qquad G_-(t) = -\frac{c(t)}{d(t)},$$

$$g_+(t) = \frac{c[\alpha(t)]h(t) - d(t)h[\alpha(t)]}{c[\alpha(t)]b(t) - a[\alpha(t)]d(t)}, \qquad g_-(t) = \frac{a[\alpha(t)]h(t) - b(t)h[\alpha(t)]}{a[\alpha(t)]d(t) - b(t)c[\alpha(t)]}.$$

Conditions (3) ensure the fulfillment of the necessary solvability conditions (5) for the problems (6) and (7) for an arbitrary function $h(t) \subset H$.

Denote $\varkappa^+ = \operatorname{Ind} G_+(t)$, $\varkappa^- = \operatorname{Ind} G_-(t)$.

Theorem 1. The homogeneous problem (1) has $k^+ + k^-$ linearly independent solutions, where

$$k^+ = \frac{1}{2}(1 - \operatorname{sign}\varkappa^+)\left(-\frac{\varkappa^+}{2} + \frac{1+\lambda_+}{2}\right),$$

$$k^- = \frac{1}{2}(1 + \operatorname{sign}\varkappa^-)\left(\frac{\varkappa^-}{2} + \frac{1+\lambda_-}{2}\right); \tag{8}$$

$$\operatorname{sign}\varkappa^+ = 1, \quad \text{if } \varkappa^+ > 0; \qquad \operatorname{sign}\varkappa^+ = -1, \quad \text{if } \varkappa^+ \leqslant 0;$$

$$\operatorname{sign}\varkappa^- = 1, \quad \text{if } \varkappa^- \geqslant 0; \qquad \operatorname{sign}\varkappa^- = -1, \quad \text{if } \varkappa^- < 0;$$

λ_+ and λ_- are the values assumed at both fixed points of $\alpha(t)$, respectively, by the coefficients $G_+(t)$ and $G_-(t)$, if the indices of these coefficients $\varkappa^+$ and $\varkappa^-$ are even numbers. If the index $\varkappa^+$ or $\varkappa^-$ is an odd number, then in formulas (8) we conventionally take the corresponding numbers λ_+ or λ_- to be equal to zero. The general solution $\{\Phi^+(z), \Phi^-(z)\}$ of the homogeneous problem (1) is given by the formulas

$$\Phi^\pm(z) = X^\pm(z)\left[P^\pm_{k^\pm}(z) + \frac{1}{2\pi i}\int_L \frac{\varphi_\pm(\tau)}{\tau - z}\,d\tau\right], \tag{9}$$

where

$$P^+_{k^+}(z) = \sum_{k=0}^{k^+} \frac{c_k}{z^k} \quad \text{and} \quad P^-_{k^-}(z) = \sum_{k=0}^{k^-} d_k z^k$$

are rational functions with arbitrary coefficients and with poles respectively at the points $z = 0$ and $z = \infty$, of orders not exceeding k^+ and k^-; $\varphi_\pm(t)$ are solutions of the Fredholm integral equations

$$K_\pm\varphi = \varphi_\pm(t) \pm \frac{1}{2\pi i}\int_L \left[\frac{1}{\tau - t} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}\right]\varphi_\pm(\tau)\,d\tau =$$

$$= \pm\left[\tilde{\lambda}_\pm P^\pm_{k^\pm}[\alpha(t)] - P^\pm_{k^\pm}(t)\right].$$

The canonical functions ([3]) $X^\pm(z)$ are defined by the formulas

$$X^\pm(z) = z^{-\varkappa^\pm/2}\exp\frac{1}{2\pi i}\int_L \frac{\xi_\pm(\tau)}{\tau - z}\,d\tau;$$

$\xi_\pm(t)$ are solutions of the integral equations

$$K_\pm\xi = -\ln G^*_\pm(t);$$

$$G^*_\pm(t) = \tilde{\lambda}_\pm\left[\frac{\alpha(t)}{t}\right]^{\varkappa^\pm/2} G_\pm(t), \quad \text{if } \varkappa^+ \text{ and } \varkappa^- \text{ are even numbers,}$$

and

$$X^{\pm}(z) = (z - t_1) z^{-(\varkappa^{\pm}+1)/2} \exp \frac{1}{2\pi i} \int_L \frac{\eta_{\pm}(\tau)}{\tau - z} \, d\tau;$$

$\eta_{\pm}(t)$ are solutions of the equations

$$K_{\pm}\eta = -\ln G^{**}_{\pm}(t);$$

$$G^{**}_{\pm}(t) = \frac{t - t_1}{\alpha(t) - t_1} \left[\frac{\alpha(t)}{t} \right]^{(\varkappa^{\pm}+1)/2} G_{\pm}(t), \quad \text{if } \varkappa^{+} \text{ and } \varkappa^{-} \text{ are odd numbers;}$$

$$G_{+}(t_1) = -1;$$

$$\tilde{\lambda}_{\pm} = \lambda_{\pm}, \quad \text{if } \varkappa^{\pm} \text{ are even numbers;} \qquad \tilde{\lambda}_{\pm} = 1, \quad \text{if } \varkappa^{\pm} \text{ are odd numbers.}$$

Corollary. The homogeneous problem (1) is unsolvable ($k^{+} = k^{-} = 0$) in the following cases: 1) $\chi^{+} > 0,\ \chi^{-} < 0$; 2) $\chi^{+} > 0,\ \chi^{-} = 0,\ \lambda_{-} = -1$; 3) $\chi^{+} = 0,\ \chi^{-} < 0,\ \lambda_{+} = -1$; 4) $\chi^{+} = \chi^{-} = 0,\ \lambda_{+} = \lambda_{-} = -1$.

Theorem 2. *The nonhomogeneous boundary-value problem* (1) *is unconditionally solvable if* $\chi^{+} \leqslant 0,\ \chi^{-} \geqslant 0$. *The general solution of the problem is given by formulas* (9), *where the functions* $P^{\pm}_{k^{\pm}}(z)$ *and* $X^{\pm}(z)$ *have the same meaning as in Theorem 1, while the functions* $\varphi_{+}(t)$ *and* $\varphi_{-}(t)$ *are solutions of the Fredholm integral equations*

$$\mathrm{K}_{+}\varphi = \tilde{\lambda}_{+} P^{+}_{k^{+}}[\alpha(t)] - P^{+}_{k^{+}}(t) + \frac{c(t)h[\alpha(t)] - d[\alpha(t)]h(t)}{c(t)b[\alpha(t)] - a(t)d[\alpha(t)]} \, \frac{1}{X^{+}(t)},$$

$$\mathrm{K}_{-}\varphi = P^{-}_{k^{-}}(t) - \tilde{\lambda}_{-} P^{-}_{k^{-}}[\alpha(t)] + \frac{b[\alpha(t)]h(t) - a(t)h[\alpha(t)]}{a(t)d[\alpha(t)] - b[\alpha(t)]c(t)} \, \frac{1}{X^{-}(t)}.$$

If $\chi^{+} > 0,\ \chi^{-} \geqslant 0$, *then* $P^{-}_{k^{+}}(z) \equiv \frac{1}{2}(1 + \tilde{\lambda}_{+})A$, *where* $A = -\frac{1}{2\pi i} \int_L \frac{\varphi_{+}(t)}{t} \, dt$, *and for the solvability of the problem when* $\tilde{\lambda}_{+} = 1$ *the conditions*

$$\int_L t^{-k} \varphi_{+}(t) \, dt = 0, \qquad k = 2, 3, \ldots, \chi^{+}_{1}*, \tag{10}$$

must be satisfied; to these, when $\tilde{\lambda}_{+} = -1$, *one more condition is added,*

$$\int_L \frac{\varphi_+(t)}{t}\,dt = 0. \qquad (10')$$

If $\chi^+ \leqslant 0,\ \chi^- < 0,$ *then* $P^-_{k^-}(z) \equiv 0,$ *and the solvability conditions*

$$\int_L t^{k-1}\varphi_-(t)\,dt = 0, \qquad k = 1, 2, \dots, -\chi_1^-, \qquad (11)$$

are required; to these, when $\tilde{\lambda}_- = -1,$ *the condition*

$$\int_L \left\{\frac{b[\alpha(t)]h(t) - a(t)h[\alpha(t)]}{a(t)d[\alpha(t)] - b[\alpha(t)]c(t)}\right\}\frac{\psi(t)}{X^-(t)}\,dt = 0, \qquad (12)$$

is added, where $\psi(t)$ *is a nontrivial solution of the equation* $\mathrm{K}'_-\psi = 0,$ *adjoint to the equation* $\mathrm{K}_-\varphi = 0.$

If $\chi^+ > 0,\ \chi^- < 0,$ *then* $P^+_{k^+}(z) \equiv \frac{1}{2}(1 + \tilde{\lambda}_+)A,\ P^-_{k^-}(z) \equiv 0,$ *and the conditions* (10), (10′), (11), (12) *must be fulfilled in the case* $\tilde{\lambda}_+ = \tilde{\lambda}_- = -1,$ *the conditions* (10), (10′), (11) *or the conditions* (10), (11), (12), *if respectively* $\tilde{\lambda}_+ = -\tilde{\lambda}_- = -1$ *or* $\tilde{\lambda}_+ = -\tilde{\lambda}_- = 1,$ *and the conditions* (10), (11) *in the remaining cases. The indicated solvability conditions are necessary and sufficient.*

Let us note that the nonhomogeneous problem (1) is unconditionally solvable and has a unique solution if: 1) $\chi^+ = \chi^- = 0,\ \lambda_+ = \lambda_- = -1$; 2) $\chi^+ = 0,\ \chi^- = -1,\ \lambda_+ = -1$; 3) $\chi^+ = 0,\ \chi^- = -2,\ \lambda_+ = -\lambda_- = -1$; 4) $\chi^+ = 1,\ \chi^- = 0,\ \lambda_- = -1$; 5) $\chi^+ = -\chi^- = 1$; 6) $\chi^+ = 1,\ \chi^- = -2,\ \lambda_- = 1$; 7) $\chi^+ = 2,\ \chi^- = 0,\ \lambda_+ = -\lambda_- = 1$; 8) $\chi^+ = 2,\ \chi^- = -1,\ \lambda_+ = 1$; 9) $\chi^+ = -\chi^- = 2,\ \lambda_+ = \lambda_- = 1$.

Theorems 1 and 2 show that in the case of indices χ^+ and χ^- of opposite sign, for the boundary-value problem (1) an assertion of Noether type is valid: from the nontrivial solvability of the homogeneous problem (1)

* By $\chi_1^\pm$ we denote $\chi^\pm/2$, if $\chi^\pm$ are even numbers, and $(\chi^\pm + 1)/2$, if $\chi^\pm$ are odd numbers.

it follows that the corresponding nonhomogeneous problem is unconditionally solvable; if, however, the homogeneous problem has only the trivial solution, then the corresponding nonhomogeneous problem is, generally speaking, unsolvable, except perhaps for the values (χ^+, χ^-): $(1, -1)$, $(2, -2)$, $(1, -2)$, $(2, -1)$. If the indices χ^+ and χ^- have one and the same sign, then for the boundary-value problem under investigation a statement of the Fredholm-alternative type holds: if the homogeneous problem is nontrivially solvable, then the corresponding nonhomogeneous problem is, generally speaking, unsolvable.

§ 2. Consider the integro-functional equation

$$A(t)\varphi(t) + B(t)\varphi[\alpha(t)] + \frac{C(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau + \frac{D(t)}{\pi i}\int_L \frac{\varphi(t)}{\tau - \alpha(t)}\,d\tau = H(t). \quad (13)$$

We shall assume that the conditions

$$\begin{gathered} A^2(t) - C^2(t) \neq 0, \qquad B^2(t) - D^2(t) \neq 0, \\ A(t)A[\alpha(t)] + C(t)C[\alpha(t)] = B(t)B[\alpha(t)] + D(t)D[\alpha(t)], \\ A(t)C[\alpha(t)] + C(t)A[\alpha(t)] = B(t)D[\alpha(t)] + D(t)B[\alpha(t)] \end{gathered} \quad (14)$$

are fulfilled.

The case where, in equation (13), $B(t) = C(t) = 0$ was considered in the author' s note ([5]).

If the assumptions (14) are fulfilled, then equation (13), by the usual method of continuation into the complex domain with the aid of an integral of Cauchy type (see, for example, ([6])), is reduced to the boundary-value problem (1) with conditions (3). The solution of this latter problem, by virtue of its representability by an integral of Cauchy type, must vanish at the point $z = \infty$. In this connection the homogeneous equation (13) will have one linearly independent solution fewer than the number of solutions indicated by Theorem 1, if $k^- \neq 0$, while the nonhomogeneous equation (13), for $\chi^- < 0$, is solvable if, in addition to the conditions indicated in the formulation of Theorem 2, the condition

$$\int_L t^{-\chi_1^- - 1}\varphi_-(t)\,dt = 0$$

is also fulfilled.

Comparing the numbers of linearly independent solutions of the homogeneous equation (13) and of the homogeneous equation adjoint to it, we arrive at the following principal result.

Theorem 3. *The index I of the integro-functional equation (13) is expressed by the formula*

$$I = \frac{\chi^- - \chi^+}{2} + \frac{\lambda_+ + \lambda_-}{2}.$$

From Theorem 3 it follows that

Theorem 4. *The index of the integro-functional equation (13) is equal to zero in one of the following cases: 1) $\chi^+ = \chi^-$, with $\lambda_+ = -\lambda_- = \pm 1$, or χ^+ and*

χ^- are odd numbers ($\lambda_+ = \lambda_- = 0$); 2) $\chi^+ - \chi^- = \pm 2$, $\lambda_+ = \lambda_- = \pm 1$; 3) $\chi^+ - \chi^- = \pm 1$, with $\lambda_+ = \pm 1$, $\lambda_- = 0$ or $\lambda_- = \pm 1$, $\lambda_+ = 0$.

Theorem 5. *The integro-functional equation (13) is normally solvable.*

Thus, if any of the conditions of Theorem 4 is fulfilled, then for the integral equation (13) all three Fredholm theorems are valid.

Rostov-on-Don
State University

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Note: Figure translations are in progress. See original paper for figures.