

THE PROBLEM OF INELASTIC SCATTERING

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1961

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Abstract

Full Text

MATHEMATICS

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THE PROBLEM OF INELASTIC SCATTERING

(Presented by Academician V. I. Smirnov on 12 V 1961)

Mathematically, the problem of inelastic scattering of particles by a scattering center can be formulated as follows ^(1,2):

It is required to find a solution of the infinite system of equations

$$(\Delta + k_n^2)\psi_n(x) + \sum_{m=0}^{\infty} c_{nm}(x)\psi_m(x) = 0 \quad (n = 0, 1, \dots), \quad (1)$$

having the form

$$\psi_0(x)e^{ik_0\mu \cdot x} + w_0(x),$$

$$\psi_n(x) = w_n(x) \quad (n > 0), \quad (2)$$

where the functions $w_n(x)$ satisfy the radiation conditions

$$|w_n(x)| = O\left(\frac{1}{|x|}\right), \quad \frac{\partial w_n(x)}{\partial |x|} - ik_n w_n(x) = O\left(\frac{1}{|x|}\right); \quad (3)$$

x is a point of three-dimensional Euclidean space E^3 ; μ is the direction of the incident plane wave $e^{ik_0\mu \cdot x}$; $w_0(x)$ is the elastically scattered wave, and $w_n(x)$ is the inelastically scattered wave with momentum k_n .

In ⁽²⁾, by the method of successive approximations, the existence and uniqueness of the solution of the inelastic scattering problem was proved in the case where the system (1) is finite and the perturbation $c_{nm}(x)$ is sufficiently small.

We shall assume that:

- 1) The numbers k_n are bounded above and below, i.e.

$$0 < m \leq k_n \leq M < +\infty.$$

- 2) $c_{nm}(x) = v(x)\delta_{nm} + v_{nm}(x)$, where $v(x)$ is a real function, and the matrix $\{v_{nm}(x)\}_0^\infty$ is Hermitian, i.e.

$$v_{nm}(x) = \overline{v_{mn}(x)}.$$

- 3) The estimates hold

$$|v(x)| \leq \frac{C}{1 + |x|^{3+1/2+\varepsilon}}, \quad |v_{nm}(x)| \leq \frac{a_{nm}}{1 + |x|^{3+1/2+\varepsilon}}, \quad \text{where } \varepsilon > 0.$$

- 4) The matrix $\{a_{nm}\}_0^\infty$ generates a completely continuous operator in l_2 . For this it is sufficient, for example, to require that

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} a_{nm} = 0.$$

Under these assumptions the following is true:

Theorem. *There exists a unique solution of the inelastic scattering problem.*

We outline the proof of this theorem. Considering $\psi_n(x)$ and $w_n(x)$ as the components of vector-functions $\Psi(x)$ and $W(x)$ with values in the Hilbert space l_2 , the system (1) can be written in the form

$$\Delta \Psi(x) + K^2 \Psi(x) + C(x) \Psi(x) = 0, \quad (4)$$

and (2) and the radiation condition (3), respectively, in the form

$$\Psi(x) = e^{iK\mu \cdot x} \varphi_0 + W(x), \quad (5)$$

$$\|W(x)\|_{l_2} = O\left(\frac{1}{|x|}\right), \quad \left\| \frac{\partial W(x)}{\partial |x|} - iKW(x) \right\|_{l_2} = o\left(\frac{1}{|x|}\right). \quad (6)$$

Here K is an operator whose matrix is diagonal, with elements k_n on the diagonal; $C(x)$ is the operator with matrix $\{c_{nm}(x)\}_0^\infty$; φ_0 is the vector with components $(1, 0, \dots)$.

According to condition 1), the operator K is positive and $(Ku, u) \geq m(u, u)$. From conditions 2), 3), and 4) it follows that $C(x)$ is a bounded self-adjoint operator in l_2 , and

$$\|C(x)\|_{l_2} \leq \frac{C}{1 + |x|^{3+1/2+\varepsilon}}, \quad \text{where } \varepsilon > 0.$$

Substituting (5) into (4) and taking (6) into account, we obtain the following integral equation for $W(x)$:

$$W(x) = \frac{1}{4\pi} \int \frac{e^{iK|x-s|}}{|x-s|} C(s) e^{iK\mu \cdot s} \varphi_0 ds + \frac{1}{4\pi} \int \frac{e^{iK|x-s|}}{|x-s|} C(s) W(s) ds. \quad (7)$$

Thus, if a solution of the inelastic scattering problem exists, then the function $W(x)$ satisfies the integral equation (7). It turns out that the converse also holds. Namely, let Φ denote the Banach space of vector-functions $\varphi(x)$ with values in l_2 , continuously depending on $x \in E^3$, $\|\varphi(x)\|_\Phi = \sup_{x \in E^3} \|\varphi(x)\|_{l_2}$, i.e. $\Phi = l_2 \otimes C(E^3)$.

Then, if there exists a unique solution of equation (7) in the space Φ , there also exists a unique solution of the scattering problem.

For brevity, denote

$$\frac{1}{4\pi} \int \frac{e^{iK|x-s|}}{|x-s|} f(s) ds = Tf. \quad (8)$$

The inelastic scattering problem is thus reduced to the study of the equation

$$u(x) = h(x) + TC(x)u(x) \quad (9)$$

in the space Φ . The operator $C(x)$, according to assumptions 2) and 4), has the form

$$C(x) = v(x)E + V(x), \quad (10)$$

where E is the identity operator, and the operator $V(x)$ is completely continuous. Therefore equation (9) can be written in the form

$$u(x) = h(x) + Tv(x)u(x) + TV(x)u(x). \quad (11)$$

On the basis of the works ^(3,4), using the diagonality of the operator $E - Tv(x)$, one can prove that the operator $[E - Tv(x)]^{-1}$ exists and is bounded in the space Φ .

If we apply the operator $[E - Tv(x)]^{-1}$ to the left- and right-hand sides of equation (11), we obtain the equivalent equation

$$u(x) = [E - Tv(x)]^{-1}h(x) + [E - Tv(x)]^{-1}TV(x)u(x) \quad (12)$$

with the completely continuous operator $[E - Tv(x)]^{-1}TV(x)$.

On the basis of Fredholm's theorem, one may assert that there exists a unique solution of equation (12) (and consequently of equation (9)) for any free term, if the homogeneous equation

$$u(x) = [E - Tv(x)]^{-1}TV(x)u(x) \quad (13)$$

has only the trivial solution $u(x) \equiv 0$.

Equation (13) in the space Φ is equivalent to the equation

$$u(x) = TC(x)u(x). \quad (14)$$

Analogously to lemma (3), one can prove that a solution of equation (14) from the space Φ automatically belongs to $l_2 \otimes \bar{L}_2(E^3)$, i.e.

$$\int \|u(x)\|_{l_2}^2 dx < +\infty.$$

But a solution of equation (14) is a solution of the differential equation (4). The absence of nontrivial solutions of equation (4) from $l_2 \otimes L_2(E^3)$ can be proved in the same way as for the ordinary Schrödinger equation (5). Thus the proof of the theorem is completed.

The solvability of the scattering problem for the Schrödinger equation with operator coefficients (4)–(6) can be proved without condition (10), assuming only that the operator $C(x)$ is self-adjoint and bounded. In doing so, however, one has to assume that $k_n \rightarrow \infty$ as $n \rightarrow \infty$, since in this case one succeeds in proving the complete continuity of the operator $[TC(x)]^2$ and in applying Fredholm's theorem to the iterated equation (9).

Such a problem is led to, for example, by the nonstationary scattering problem for the equation

$$\left[\Delta - \frac{\partial^2}{\partial t^2} + c(x, t) \right] u(x, t) = 0 \quad (15)$$

with a periodic perturbation $c(x, t)$ (6).

Indeed, let $c(x, t) = c_1(x) + c_2(x) \sin \omega t$, and let the incident plane wave $e^{i\omega_0(\mu \cdot x - t)}$ have frequency ω_0 .

If one seeks a solution of equation (15) in the form

$$u(x, t) = \sum_{n=-\infty}^{+\infty} u_n(x) e^{-i(\omega_0 + n\omega)t},$$

then for the functions $u_n(x)$ we obtain the system of equations

$$\Delta u_n(x) + (\omega_0 + n\omega)^2 u_n(x) + c_1(x)u_n(x) + \frac{c^2(x)}{2i} [u_{n-1} - u_{n+1}] = 0. \quad (16)$$

The scattering problem in this case will consist in finding solutions of the system (16) having the form

$$u_0(x) = e^{i\omega_0 \mu \cdot x} + w_0(x),$$

$$u_n(x) = w_n(x), \quad n \neq 0, \quad (17)$$

where $w_n(x)$ satisfy the radiation conditions

$$|w_n(x)| = O\left(\frac{1}{|x|}\right), \quad \frac{\partial w_n(x)}{\partial |x|} - i|\omega_0 + n\omega|w_n(x) = o\left(\frac{1}{|x|}\right). \quad (18)$$

Let the functions $c_1(x)$ and $c_2(x)$ be real and be majorized by

$$\frac{C}{1 + |x|^{3+1/2+\varepsilon}},$$

where $\varepsilon > 0$. In addition, let ω_0 not be a multiple of ω , i.e. $\omega_0 \neq n\omega$. Then, on the basis of the preceding, one can assert that there exists a unique solution of the nonstationary scattering problem under a periodic perturbation.

In conclusion, the author expresses his gratitude to Yu. M. Berezanskii for posing the problem and for valuable comments.

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Received
11 V 1961

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