MATHEMATICS

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1961

SovietRxiv

Abstract

Full Text

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ON DISTORTION UNDER QUASICONFORMAL MAPPINGS

(Presented by Academician I. G. Petrovskii, 2 XII 1960)

1. In this note we consider the class of Q-quasiconformal mappings and classes of quasiconformal mappings in the mean, introduced in ([1]).

By a Q-quasiconformal mapping $W = T(z)$ of a plane simply connected domain D onto a similar domain Δ we shall mean an orientation-preserving topological correspondence of these domains for which the inequalities

$$Q^{-1} \operatorname{mod} \Omega(z) \leqslant \operatorname{mod} T\{\Omega(z)\} \leqslant Q \operatorname{mod} \Omega(z),$$

hold, where $\Omega(z)$ is an arbitrary quadrilateral contained in D together with its boundary.

The fundamental theorem of M. A. Lavrent'ev, establishing the equicontinuity of Q-quasiconformal mappings in simply connected domains with Jordan boundary, also gave an estimate of the distortion of the distance between points of the domain in modulus. Subsequent authors, refining this estimate, brought it to the form ([2]): if $w = T(z)$ is a Q-quasiconformal mapping of the disk $|z| < 1$ onto the disk $|w| < 1$, $T(0) = 0$, then the inequality

$$|T(z_1) - T(z_2)| < A|z_1 - z_2|^{1/Q},$$

holds, sharp in the sense of order. The constant A is such that $16^{1-1/Q} \leqslant A < 16$.

The derivation of this inequality was essentially based on the known upper estimate for the Teichmüller function $\varphi_Q(z)$ in the inequality $|T(z)| \leqslant \varphi_Q(|z|)$, sharp in the sense that for any point z_0, $|z_0| < 1$, there exists a Q-quasiconformal mapping $\widetilde{T}(z)$ of the disk $|z| < 1$ onto itself for which $|\widetilde{T}(z_0)| = \varphi_Q(|z_0|)$. This estimate is known:

$$\varphi_Q(|z|) < 4|z|^{1/Q}.$$

However, quite recently the remarkable inequality ([3]) was obtained

$$|T(z)| \leqslant 4^{1-1/Q}|z|^{1/Q}.$$

Comparing it with the preceding estimate, we obtain

$$\varphi_Q(|z|) \leqslant 4^{1-1/Q}|z|^{1/Q}.$$

Carrying out the proof in ([2]) with the new estimate of the function $\varphi_Q(r)$, we obtain the final form of M. A. Lavrent' ev' s theorem:

Theorem (on distortion of moduli). *If $w = T(z)$ is a Q-quasiconformal mapping of the disk $|z| < 1$ onto the disk $|w| < 1$, $T(0) = 0$, then for*

for any points z_1, z_2 from $|z| < 1$ the inequality holds

$$16^{-Q+1}|z_1 - z_2|^Q \leq |T(z_1) - T(z_2)| \leq 16^{1-1/Q}|z_1 - z_2|^{1/Q}. \tag{1}$$

Neither the order nor the constant in (1) can be improved.

Along with the theorem stated above, there are estimates for the distortion of the magnitudes of angles under Q-quasiconformal mappings.

Theorem 1. Under a Q-quasiconformal mapping of the disk $|z| < 1$ onto itself, the image of a path nontangent to the unit circle and ending at some point on it remains a path nontangent to the unit circle ([4]), and the magnitudes α, β of the corresponding angles between these paths and the unit circle are related by the inequality

$$(4\pi)^{-Q+1}\left(\frac{\pi}{4}\right)^Q \alpha^Q < \beta < (4\pi)^{1-1/Q}\frac{4}{\pi}\alpha^{1/Q}; \tag{2}$$

in this inequality the power order with exponent $1/Q$ is sharp.

For the case of a Q-quasiconformal mapping of the strip $0 \leq \operatorname{Im} z \leq 1$ onto itself preserving the points $\pm\infty$, this theorem is a refinement of the corresponding result in ([4]), and its proof in this case rests on inequality (1) and simple estimates for conformal mappings.

2. Let us now consider the classes of quasiconformal mappings in the mean introduced in ([1]). Let $\zeta = \zeta(z)$ be a differentiable mapping of the disk $|z| < 1$ onto itself with nonzero Jacobian. For arbitrary real $m \geq 1$ define

$$\operatorname{Im}(\zeta) = \frac{1}{\pi}\iint_{|\zeta|<1}\left(\frac{|p|^2 + |q|^2}{|p|^2 - |q|^2}\right)^m d\xi\, d\eta,$$

where $p(z)$ and $q(\bar{z})$ are the complex derivatives of the function $\zeta = \zeta(z)$, considered at the point $z(\zeta)$, $\zeta = \xi + i\eta$.

Following Ahlfors ([1]), we shall say that the mapping $\zeta = \zeta(z)$ belongs to the class $Q_m(K)$ if $\text{Im}(\zeta) \leq K^m$ for some $K \geq 1$. The closure of the class $Q_m(K)$ with respect to uniform convergence on compact subsets in $|z| < 1$ will be denoted by $\overline{Q}_m(K)$.

Ahlfors proved the equicontinuity in $|z| < 1$ of mappings of the classes $\overline{Q}_m(K)$, $m > 2$. By modifying this proof, one can obtain the equicontinuity of mappings of the classes $\overline{Q}_m(K)$, $m > 1$, and an estimate for the distortion.

Theorem 2. If $\zeta = \zeta(z) \in \overline{Q}_m(K)$ for $m > 1$, then for any points z_1, z_2 from the disk $|z| \leq \rho$, $0 \leq \rho < 1$, there is a constant B, depending on ρ and m, such that the inequality

$$\left[\left|\ln \frac{1}{|\zeta_1 - \zeta_2|}\right|\right]^{-(m-1)} < B \left[\left|\ln \frac{1}{|z_1 - z_2|}\right|\right]^{-m}, \tag{3}$$

holds, where $\zeta_i = \zeta(z_i)$, $i = 1, 2$.

Naturally, the estimate in (3) cannot be considered definitive.

The author expresses his gratitude to A. I. Markushevich, under whose supervision the note was written, and also to B. V. Shabat for his attention to this work.

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Received
1 XII 1960

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Note: Figure translations are in progress. See original paper for figures.