V. I. ARNOLD

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Abstract

Full Text

V. I. ARNOLD

ON THE BIRTH OF CONDITIONALLY PERIODIC MOTION FROM A FAMILY OF PERIODIC MOTIONS

(Presented by Academician A. N. Kolmogorov, 9 XII 1960)

§ 1. A motion of a point x, y on a torus T is called **conditionally periodic** if

$$\frac{dy}{dx} = \lambda, \tag{1}$$

where λ is an irrational constant, and x and y are coordinates on the torus, so that $(x+k, y+l)$ and (x, y) are one and the same point of T. Consider a family of close differential equations

$$\frac{dy}{dx} = \lambda + a + \varepsilon f(x, y), \tag{2}$$

where a and ε are parameters, and $f(x, y)$ is an analytic function.

Recently ([1]) I showed that if the "perturbation" $\varepsilon f(x, y)$ is small, then there exists such an $a(\varepsilon)$ that equation (2), for $a = a(\varepsilon)$, can be reduced to the form (1) by an analytic change of variable.

In the present note the degenerate case $\lambda = 0$ is considered; the unperturbed motion takes place along the parallels of the torus $y = \text{const}$, i.e. it is periodic. It turns out that under many small perturbations such a motion passes into a conditionally periodic one. It is of interest to trace this transition because systems close to degenerate ones often occur in mechanics.

The difficulties that arise in reducing the equation to the form (1), owing to the presence of small denominators, are overcome by means of successive approximations of Newtonian type. A. N. Kolmogorov ([2]) first applied this method in constructing conditionally periodic motions of a system with Hamiltonian function

$$H(p, q) = H_0(p_1, \dots, p_n) + \varepsilon \widetilde{H}(p_1, \dots, p_n, q_1, \dots, q_n) + \cdots.$$

What is essential in the present note is the introduction of new (though simple) arithmetical facts; their appearance is connected with the degeneracy of the unperturbed system. By generalizing Theorem 2 of our note, one may extend the results of ([2]) to systems with Hamiltonian function

$$H = H_0(p_1, \dots, p_k) + \varepsilon H_1(p_1, \dots, p_n, q_1, \dots, q_k) +$$

$$+\varepsilon^2 \widetilde{H}(p_1, \dots, p_n, q_1, \dots, q_n) + \cdots \quad (k < n).$$

§ 2. Denote by Λ_θ the set of points λ such that

$$|\lambda n + m| > \theta |\lambda| n^{-2}$$

for all integers m and n, $n \neq 0$. Let Λ denote the union of the sets Λ_θ for all $\theta > 0$. The following is easily proved.

Lemma. *Zero is a limit point of the set Λ_θ $(0 < \theta < 0.25)$ and a point of density of the set Λ.*

Theorem 1. *Let on the torus T there be given a differential equation*

$$\frac{dy}{dx} = \varepsilon f(x, y), \tag{3}$$

depending on the parameter ε, with analytic function $f(x, y)$. Let

$$\int_0^1 f(x, y)\, dx > 0$$

for all y. Then for every sufficiently small $\lambda \in \Lambda_0$ there is an $\varepsilon(\lambda)$ and a change of variable $z = z_\lambda(x, y)$, analytic in x, y, such that equation (3) takes the form $dz/dx = \lambda$. The set $\varepsilon(\lambda)$ $(\lambda \in \Lambda)$ has positive measure; zero is its point of density.

The proof of Theorem 1 is based on the fact that first a change of variable is made, as recommended by the usual asymptotic methods, and then

Theorem 2. *The conclusion of Theorem 1 is valid for the differential equation on the torus*

$$\frac{dy}{dx} = \varepsilon c + \varepsilon^2 F(x, y, \varepsilon), \tag{4}$$

where $c > 0$ is a constant and the function $F(x, y, \varepsilon)$ is analytic.

§ 3. Let us show how to reduce equation (3) to the form (4). In $f(x, y)$ we separate the averaged part

$$\bar{f}(y) = \int_0^1 f(\xi, y)\, d\xi$$

and the variable part

$$\tilde{f}(x, y) = f(x, y) - \bar{f}(y).$$

First introduce on the torus such a coordinate

$$y_1 = y_1(x, y, \varepsilon) = y + \varepsilon h(x, y), \tag{5}$$

so that the variable part of dy_1/dx is of order ε^2. Obviously,

$$\frac{dy_1}{dt} = \varepsilon \bar{f} + \varepsilon \left(\tilde{f} + \frac{\partial h}{\partial x} \right) + \varepsilon^2 f \frac{\partial h}{\partial y},$$

therefore in (5) one must put

$$h(x, y) = -\int_0^x \tilde{f}(\xi, y)\, d\xi.$$

Then

$$\frac{dy_1}{dx} = \varepsilon \bar{f}(y_1) + F_1(x, y_1, \varepsilon), \tag{6}$$

where

$$F_1 = \varepsilon[\bar{f}(y_1) - \bar{f}(y)] + \varepsilon^2 f \frac{\partial h}{\partial y}.$$

Since

$$\bar{f}(y_1) - \bar{f}(y) = -\varepsilon \bar{f}' \int_0^x \tilde{f}(\xi, y)\, d\xi,$$

the function F_1 in (6) contains the factor ε^2.

Now transform the coordinate y_1 so that the term of order ε on the right-hand side becomes constant. Since, by the condition of Theorem 1, $\bar{f} > 0$, we may put

$$y_2(y_1) = \int_0^{y_1} \frac{c\,d\xi}{\bar{f}(\xi)}, \qquad \frac{1}{c} = \int_0^1 \frac{dy}{\bar{f}(y)}.$$

The constant c is determined by the requirement $y_2(1) - y_2(0) = 1$. Now equation (3) takes the form

$$\frac{dy_2}{dt} = \varepsilon c + F_2(x, y_2, \varepsilon),$$

where

$$F_2 = y_2' F_1(x, y_1(y_2), \varepsilon)$$

contains the factor ε^2, and

$$c = \left[\int_0^1 \frac{dy}{\int_0^1 f(x,y)\,dx}\right]^{-1}.$$

§ 4. Passing to the proof of Theorem 2, let us outline the first step in constructing the variable $z(x, y) = y + k(x, y)$. Obviously,

$$\frac{dz}{dx} = \lambda + \left(\varepsilon^2 F + \frac{\partial k}{\partial x} + \lambda \frac{\partial k}{\partial y}\right) + \varepsilon^2 F \frac{\partial k}{\partial y} \qquad (\lambda = \varepsilon c), \tag{7}$$

therefore in the first approximation we determine k from the condition

$$\varepsilon^2 F + \frac{\partial k}{\partial x} + \lambda \frac{\partial k}{\partial y} = \varepsilon^2 F_{00}.$$

The Fourier coefficients of the function

$$k(x, y) = \sum_{m^2+n^2 \neq 0} k_{mn} e^{2\pi i(mx+ny)}$$

are expressed in terms of the Fourier coefficients of the function $F(x, y)$:

$$2\pi k_{mn} = \frac{i\varepsilon^2 F_{mn}}{m + \lambda n}.$$

Fix ε and suppose that the rotation number [(3)] of equation (4) is $\lambda \in \Lambda_\theta$. Then $k(x, y)$ and its derivatives are of order $\varepsilon^2 \left|\frac{F}{\lambda}\right|$, i.e., of order ε. Equation (7) gives

$$\frac{dz}{dx} = \lambda + \varepsilon^2 F \frac{\partial k}{\partial y} + \varepsilon^2 F_{00}.$$

Since $\varepsilon^2 F\, \partial k/\partial y$ is of order ε^3, and the rotation number of equation (7) is equal to λ, it follows that $\varepsilon^2 F_{00}$ is of order ε^3. Thus, in the new coordinates the "perturbation" $\varepsilon^2(F_{00} + F\, \partial k/\partial y)$ is of order $\varepsilon^3 = (\varepsilon^2)^{1\frac{1}{2}}$, which ensures rapid convergence of the successive approximations.

§ 5. The constructions of § 4 make it possible to prove the following:

Main lemma. *Let the differential equation*

$$\frac{dy}{dx} = \lambda + F(x, y) \tag{8}$$

on the torus T have rotation number $\lambda \in \Lambda_\theta$, and let the function $F(x, y)$ be analytic for $|\operatorname{Im} x, y| < \rho$, with $|F| < M$. Suppose that, for some $\delta > 0$, the inequalities

$$\delta < 0.1\rho; \qquad \delta < 2^{-7\theta}; \qquad M < \delta^4 |\lambda| \tag{9}$$

are satisfied. Then there exists a change of variable $y = y(x, z)$, analytic for $|\operatorname{Im} x, z| < \rho_1 = \rho - 3\delta$, such that

$$\frac{dz}{dx} = \lambda + F_{\text{new}}(x, z),$$

where the function F_{new} is analytic for $|\operatorname{Im} x, z| < \rho_1$ and

$$|F_{\text{new}}| < M_1 = \frac{M^2}{\delta^4 |\lambda|}.$$

Under the assumptions of Theorem 2 there exist $c_0 > 0$, $\varepsilon_0 > 0$, $\rho > 0$, $N > 0$ such that, for $|\varepsilon| < \varepsilon_0$, $|\operatorname{Im} x, y| < \rho$, the function $F(x, y, \varepsilon)$ is analytic, $|F| < N$, and equation (4) has rotation number $\lambda(\varepsilon) > c_0 \varepsilon$ ($\varepsilon > 0$).

Fix θ, $0 < \theta < 0.25$; in any neighborhood of zero there are points $\lambda \in \Lambda_\theta$. Let $\lambda = \delta^{20}$ be such a point, and moreover

$$\delta < c_0, \qquad \delta^{19} < \varepsilon_0, \qquad \delta < 0.1\rho, \qquad \delta < 2^{-7\theta}, \qquad \delta < N^{-1/2}. \tag{10}$$

Then $\varepsilon = \varepsilon(\lambda) < \lambda c_0^{-1} < \delta^{19}$, and we are in the conditions of the main lemma with $M = \delta^{36} > N\varepsilon^2$. In view of conditions (10), inequalities (9) are satisfied; therefore, by the main lemma,

$$M_1 = \delta^{48} = M^{1\frac{1}{3}}.$$

Applying the main lemma s times, we obtain $M_s = M^{(1\frac{1}{3})^s}$; hence proving the convergence of the approximations as $s \to \infty$ presents no difficulties.

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

[1] V. I. Arnold, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **25**, No. 1, 21 (1961).
[2] A. N. Kolmogorov, *DAN*, **98**, No. 4, 527 (1954).
[3] A. Poincaré, *On Curves Defined by Differential Equations*, Moscow, 1947, Ch. XV.

Note: Figure translations are in progress. See original paper for figures.