MATHEMATICS

Academician B. S. STECHKIN and S. B. STECHKIN

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

Academician B. S. STECHKIN and S. B. STECHKIN

QUADRATIC MEAN AND ARITHMETIC MEAN

Let $y(x) \in L^2[0,1]$. Put

$$\varphi_0(x) = y(x), \qquad y_0 = \int_0^1 |\varphi_0(x)|\,dx,$$

$$\varphi_k(x) = |\varphi_{k-1}(x)| - y_{k-1}, \qquad y_k = \int_0^1 |\varphi_k(x)|\,dx \qquad (k = 1, 2, \ldots).$$

Then the formula

$$\int_0^1 y^2(x)\,dx = \sum_{k=0}^{\infty} y_k^2 = \sum_{k=0}^{\infty} \left\{ \int_0^1 |\varphi_k(x)|\,dx \right\}^2 . \tag{1}$$

is valid.

This formula relates the quadratic mean of the function $y(x)$ to the arithmetic means of the functions $|\varphi_k(x)|$, which depend very simply on $y(x)$.

Applying the identity

$$\int_0^1 \varphi^2(x)\,dx = \left\{ \int_0^1 \varphi(x)\,dx \right\}^2 + \int_0^1 \left\{ \varphi(x) - \int_0^1 \varphi(t)\,dt \right\}^2 dx$$

to the functions $|\varphi_k(x)|$ $(k = 0, 1, \ldots, n-1)$, we infer that

$$\int_0^1 \varphi_k^2(x)\,dx = y_k^2 + \int_0^1 \varphi_{k+1}^2(x)\,dx \qquad (k = 0, 1, \ldots, n-1),$$

whence

$$\int_0^1 y^2(x)\,dx = \sum_{k=0}^{n-1} y_k^2 + \int_0^1 \varphi_n^2(x)\,dx; \tag{2}$$

$$\sum_{k=0}^{\infty} y_k^2 \leq \int_0^1 y^2(x)\,dx. \tag{3}$$

It remains to show that for every function $y(x)$ this inequality becomes an equality, i.e., that

$$\int_0^1 \varphi_n^2(x)\,dx \to 0 \qquad (n \to \infty). \tag{4}$$

For this we shall need two lemmas.

Lemma 1. Let $p \geq 0$, and on a set E, $\operatorname{mes} E = \delta > 0$, the inequality $|\varphi_p(x)| \geq M$ be satisfied. Then

$$K_p = \sum_{k=p}^{\infty} y_k \geq M. \tag{5}$$

Proof. Suppose that $K_p < M$, and let us derive a contradiction. On the set E we have

$$\varphi_{p+1}(x) = |\varphi_p(x)| - y_p \geq M - y_p \geq M - K_p > 0,$$

$$\varphi_{p+2}(x) = |\varphi_{p+1}(x)| - y_{p+1} \geq M - y_p - y_{p+1} \geq M - K_p > 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\varphi_n(x) \geq M - K_p > 0 \qquad (n = p+1,\ p+2, \ldots).$$

Hence

$$y_n = \int_0^1 |\varphi_n(x)|\,dx \geq \int_E |\varphi_n(x)|\,dx \geq \delta(M - K_p) \qquad (n = p+1,\ p+2, \ldots),$$

which contradicts the condition $y_n \to 0$ $(n \to \infty)$ (see (3)). Inequality (5) is proved.

Lemma 2. Let $0 \leq p \leq n$. Put $\varepsilon_p = \max_{m>p} y_m$.

1) If at the point $x_0 \in [0, 1]$

$$|\varphi_p(x_0)| \geq \sum_{k=p}^{n} y_k,$$

then

$$|\varphi_{n+1}(x_0)| = |\varphi_p(x_0)| - \sum_{k=p}^{n} y_k.$$

2) If at the point $x_0 \in [0,1]$

$$|\varphi_p(x_0)| < \sum_{k=p}^{n} y_k,$$

then

$$|\varphi_{n+1}(x_0)| \leq \varepsilon_p.$$

Proof. 1) We have

$$\varphi_{p+1}(x_0) = |\varphi_p(x_0)| - y_p \geq \sum_{k=p+1}^{n} y_k \geq 0,$$

$$\varphi_{p+2}(x_0) = |\varphi_{p+1}(x_0)| - y_{p+1} = |\varphi_p(x_0)| - (y_p + y_{p+1}) \geq \sum_{k=p+1}^{n} y_k,$$

…………………………………

$$\varphi_{n+1}(x_0) = |\varphi_n(x_0)| - y_n = \varphi_n(x_0) - y_n = |\varphi_p(x_0)| - \sum_{k=p}^{n} y_k \geq 0,$$

i.e.

$$|\varphi_{n+1}(x_0)| = |\varphi_p(x_0)| - \sum_{k=p}^{n} y_k.$$

2) Let us find a number m $(p \leq m \leq n)$ such that

$$\sum_{k=p}^{m-1} y_k \leq |\varphi_p(x_0)| < \sum_{k=p}^{m} y_k.$$

Then, according to part 1),

$$|\varphi_m(x_0)| = |\varphi_p(x_0)| - \sum_{k=p}^{m-1} y_k$$

Next we have

$$\varphi_{m+1}(x_0) = |\varphi_m(x_0)| - y_m = |\varphi_p(x_0)| - \sum_{k=p}^{m} y_k < 0,$$

$$|\varphi_{m+1}(x_0)| \leqslant y_m \leqslant \varepsilon_p,$$

$$\varphi_{m+2}(x_0) = |\varphi_{m+1}(x_0)| - y_{m+1} \leqslant |\varphi_{m+1}(x_0)| \leqslant \varepsilon_p,$$

$$\varphi_{m+2}(x_0) \geqslant -y_{m+1} \geqslant -\varepsilon_p,$$

i.e.

$$|\varphi_{m+2}(x_0)| \leqslant \varepsilon_p, \dots, |\varphi_{n+1}(x_0)| \leqslant \varepsilon_p,$$

and the lemma is proved.

We pass to the proof of relation (4). Consider two cases.

1st case. $\sup_x |\varphi_0(x)| = M_0 < \infty$*. Then, obviously, for any $p \geqslant 0$

$$\sup_x |\varphi_p(x)| = M_p < \infty.$$

Fix $\varepsilon > 0$, and let p be so large that

$$\varepsilon_p = \max_{m \geqslant p} y_m \leqslant \varepsilon.$$

According to Lemma 1 we can find a number $n = n(\varepsilon)$ for which

$$\sum_{k=p}^{n} y_k \geqslant M_p - \varepsilon_p.$$

Then, by Lemma 2, for almost all x for which

$$|\varphi_p(x)| \geqslant \sum_{k=p}^{n} y_k,$$

the inequality

$$|\varphi_{n+1}(x)| = |\varphi_p(x)| - \sum_{k=p}^{n} y_k \leqslant M_p - (M_p - \varepsilon_p) = \varepsilon_p$$

holds, and for all x for which

$$|\varphi_p(x)| < \sum_{k=p}^{n} y_k,$$

the inequality

$$|\varphi_{n+1}(x)| \leqslant \varepsilon_p$$

holds. Hence

$$\sup_x |\varphi_{n+1}(x)| \leqslant \varepsilon_p \leqslant \varepsilon,$$

$$r_{n+1}^2 = \int_0^1 \varphi_{n+1}^2(x)\, dx \leqslant \varepsilon^2.$$

Since, moreover,

$$r_{n+1}^2 = y_{n+1}^2 + r_{n+2}^2 \geqslant r_{n+2}^2,$$

it follows that $r_n^2 \to 0$ $(n \to \infty)$. We note that in this case $M_n \to 0$ $(n \to \infty)$, and for any $n \geqslant 0$

$$\int_0^1 y^2(x)\, dx = \sum_{k=0}^{n} y_k^2 + \theta_n M_n^2, \qquad (0 \leqslant \theta_n \leqslant 1).$$

[*] Here and below $\sup_x |\varphi(x)|$ denotes the essential supremum of the function $|\varphi(x)|$ on the interval $[0, 1]$.

2nd case. $\sup_x |\varphi_0(x)| = \infty$. Then for any $p \geqslant 0$

$$\sup_x |\varphi_p(x)| = \infty.$$

Fix $\varepsilon > 0$ and find $p \geqslant 0$ and $N > 0$ such that

$$\varepsilon_p \leqslant \varepsilon, \qquad \int_{|\varphi_p| \geqslant N} \varphi_p^2(x)\,dx \leqslant \varepsilon^2.$$

By Lemma 1, for all sufficiently large n we have

$$K_{p,n} = \sum_{k=p}^{n} y_k > N.$$

Hence, as above,

$$\begin{aligned} r_{n+1}^2 &= \int_0^1 \varphi_{n+1}^2(x)\,dx \\ &= \int_{|\varphi_p| \geqslant K_{p,n}} \varphi_{n+1}^2(x)\,dx + \int_{|\varphi_p| < K_{p,n}} \varphi_{n+1}^2(x)\,dx \\ &\leqslant \int_{|\varphi_p| \geqslant N} \varphi_p^2(x)\,dx + \varepsilon_p^2 \leqslant 2\varepsilon^2, \end{aligned}$$

and, consequently, $r_n^2 \to 0$ $(n \to \infty)$.

Thus, in both cases

$$\int_0^1 \varphi_n^2(x)\,dx \to 0 \qquad (n \to \infty),$$

and formula (1) is proved.

Received
14 XII 1960

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.