B. G. MOISHEZON

MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

B. G. MOISHEZON

ON PROJECTIVE EMBEDDINGS OF ALGEBRAIC VARIETIES

(Presented by Academician P. S. Novikov, 16 VI 1961)

MATHEMATICS

The well-known theorem of Kodaira (1) asserts that if a divisor D on a compact Kähler manifold V is such that in the class of differential forms corresponding to the cohomology class dual to it there is found a form of type $(1,1)$,

$$\sum h_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

such that the matrix $\|h_{\alpha\beta}\|$ is positive definite at every point of the manifold V, then the mapping of the manifold V into projective space corresponding to a sufficiently high multiple of the divisor D is a biregular embedding, i.e., it realizes the manifold V as a certain projective algebraic variety.

The question arises: what corresponds to this theorem in algebraic geometry?

Nakai (2) showed that, in order that a sufficiently high multiple of a divisor D on a nonsingular algebraic surface V_2 correspond to a biregular embedding of the surface V_2 in projective space, it is necessary and sufficient that the following conditions be satisfied:

$$D_2 > 0, \qquad D \cdot C > 0,$$

where C runs through all one-dimensional subvarieties of the surface V_2.

We shall assume that the ground field k is algebraically closed of characteristic zero. Let U_r be a nonsingular complete abstract algebraic variety of dimension r.

Theorem 1. Let D be a divisor on U_r such that $D^r > 0$ and $D^i \cdot C_i > 0$, where $i = 1, \dots, r-1$, and C_i runs through all subvarieties of the variety U_r having dimension i. Then, for sufficiently large n, the mapping of the variety U_r into projective space corresponding to the complete linear system $|nD|$ is a biregular embedding.

The following corollaries follow from Theorem 1:

Corollary 1. In order that a nonsingular complete abstract algebraic variety U_r be projective, it is necessary and sufficient that there exist on U_r a divisor D such that

$$D^r > 0, \qquad D^i \cdot C_i > 0 \quad (i = 1, \dots, r-1)$$

for all i-dimensional subvarieties C_i of the variety U_r.

Corollary 2. Let D be a divisor on a nonsingular algebraic variety $U_r(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$. Then, in order that in the class of differential forms corresponding to the dual–

to the divisor D in the cohomology class, there is found a form of type $(1, 1)$

$$\sum h_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

such that the matrix $\|h_{\alpha\beta}\|$ is positive definite at each point of the variety $U_r(C)$, it is necessary and sufficient that the conditions

$$D^r > 0, \qquad D^i \cdot C_i > 0 \quad (i = 1, \dots, r-1)$$

hold for all i-dimensional subvarieties C_i of the variety $U_r(C)$.

Induction on the dimension is not directly applicable to the proof of Theorem 1. Therefore one has to prove a more general theorem, from which Theorem 1 is obtained as a special case:

Theorem 2. *Let V_s be a complete abstract (possibly reducible) algebraic variety.*

Suppose that there exists a regular mapping $f : V_s \to U_r$ such that to each point of the image fV_s there corresponds only a finite number of points on V_s. Suppose that a divisor C on U_r is such that $D^s \cdot fV_s > 0$, $D^i \cdot fC_i > 0$ $(i = 1, \dots, s-1)$ for all i-dimensional subvarieties C_i of the variety V_s (where fC_i denotes the image of C_i under the mapping f). To the divisor nD there corresponds the fibred space of lines $\{nD\}$ over U_r, inducing a certain fibred space over V_s. Denote the latter by $\{nD\}_{V_s}$.

Then, for sufficiently large n, the mapping of the variety V_s into projective space, given by the group of sections of the fibred space $\{nD\}_{V_s}$, is a biregular embedding.

We give a brief outline of the proof of Theorem 2. The passage from $s - 1$ to s is carried out in the following way. First the assertion of Theorem 2 is verified for normal V_s, then for irreducible ones, and after that for reducible ones.

An important role in the passage from normal varieties to irreducible ones is played by the following consideration. If V_s^* is the normalization of the irreducible variety V_s, then on V_s there exists such a positive divisor E that, if by E^* we denote its inverse image on V_s^*, by

$$\overline{O_{V_s^*}(-E^*)}$$

the sheaf over V_s^* whose stalk at a point $P \in V_s^*$ consists of functions F such that $(F) - E^* > 0$ locally at the point P, and by

$$\overline{O^*_{V_s}(-E^*)}$$

the direct image of the sheaf $O_{V_s^*}(-E^*)$ corresponding to the mapping $V_s^* \to V_s$, then there is an exact sequence

$$0 \to \overline{O_{V_s^*}(-E^*)} \to O_{V_s} \to K \to 0,$$

where O_{V_s} is the sheaf of local rings of the variety V_s, and K is a certain coherent sheaf over V_s, concentrated on the support of the divisor E. With the help of an analogous consideration one makes the passage from irreducible varieties to reducible ones.

After it has been proved that the mapping f_n, given by the group of sections of the fibred space $\{nD\}_{V_s}$, is regular and maps distinct points of V_s to distinct points of projective space, we argue as follows. Let $V_s^{(n)}$ be the image of V_s under the mapping f_n. $V_s^{(n)}$ is a projective algebraic variety. Obviously, f_n realizes a homeomorphism (in the Zariski topology) between V_s and $V_s^{(n)}$. Denote by $\bar{O}_{V_s}$ the image of the sheaf O_{V_s} over $V_s^{(n)}$. The sheaf obtained is coherent. Denote by $\bar{O}_{V_s}(m)$ the sheaf obtained from $\bar{O}_{V_s}$ by m-fold application of Serre' s operation $F(m)$ (see (3)). It is not hard to see that the sheaf corresponding to the sheaf $\bar{O}_{V_s}(m)$ over V_s coincides with the sheaf

local sections of the fibered space $\{mnD\}_{V_s}$. Let $\alpha \in O_{V_s}(P)$ ($O_{V_s}(P)$ is the local ring of the point $P \in V_s$); y is a section of the fibered space $\{mnD\}_{V_s}$, not equal to zero at P; $y_1, \dots, y_N$ is a basis of the group of sections of the fibered space $\{mnD\}_{V_s}$, and let $\bar{\alpha}, \bar{y}, \bar{y}_1, \dots, \bar{y}_N$ be the corresponding elements over $V_s^{(n)}$. By the well-known theorem of Serre ([3]), for sufficiently large m the coherent sheaf $F(m)$ over the projective algebraic variety V is generated by the elements of the group $H^0(V, F(m))$. Hence, if m is sufficiently large, then $\bar{\alpha}\bar{y} = \sum \beta_i \bar{y}_i$, where β_i belong to the local ring $O_{V_s^{(n)}}(f_{nP})$ (f_{nP} is the image of the point P on $V_s^{(n)}$). Hence

$$\bar{\alpha} = \sum \beta_i \frac{\bar{y}_i}{\bar{y}}.$$

Consider the mapping f_{mn}. There exists a regular mapping $t_{m,n}$ such that $t_{m,n} f_{mn} = f_n$:

$$\begin{array}{ccccc} & & V_s^{(mn)} & & \\ & f_{mn} \swarrow & & \searrow t_{m,n} & \\ V_s & & \xrightarrow{f_n} & & V_s^{(n)} \end{array}$$

and, consequently, β_i belong to the local ring $O_{V_s^{(mn)}}(f_{mn}P)$.

Obviously, $\dfrac{\bar{y}_1}{\bar{y}}, \ldots, \dfrac{\bar{y}_N}{\bar{y}}$ may be regarded as nonhomogeneous coordinates in a certain affine neighborhood of the point $f_{mn}P$. It follows that $\alpha \in O_{V_s^{(mn)}}(f_{mn}P)$, i.e. the sheaves $\bar{O}_{V_s}$ and $O_{V_s^{(mn)}}$ coincide. This proves that the mapping f_{mn} is biregular.

I express my deep gratitude to my adviser I. I. Pyatetskii-Shapiro.

Received8 VI 1961

References Cited

[1] K. Kodaira, Ann. of Math., **60**, 28 (1954).
[2] Y. Nakai, J. Sci. Hirosima Univ., Ser. A, **24**, No. 1, Juli (1960).
[3] J.-P. Serre, Ann. Math., **61**, 197 (1955).

Note: Figure translations are in progress. See original paper for figures.