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Abstract

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B. A. Vostretsov and M. A. Kreines

On the Approximation of Continuous Functions by Superpositions of Plane Waves

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Let us call a sequence of directions, determined by vectors $l_i = (a_i) \neq 0,\ a_i = (a_{i1}, a_{i2}, \dots, a_{in}),\ a_{ij}$ real numbers, $i = 1, 2, \ldots$, **basic** if, for some domain D of n-dimensional Euclidean space and for any function $f(x)$, $x = (x_1, x_2, \dots, x_n)$, continuous in this domain, there exist functions $\varphi_{ik}(t_i)$, $i = 1, 2, \dots, k$, each continuous in its own interval

$$\inf_{x\in D}(a_i x) < t_i < \sup_{x\in D}(a_i x); \qquad a_i x = a_{i1}x_1 + \ldots + a_{in}x_n, \quad k = 1, 2, \dots,$$

such that the sequence of sums

$$\Phi_k(x) = \sum_{i=1}^{k} \varphi_{ik}(a_i x) \tag{1}$$

converges uniformly inside D to the function $f(x)$.

In the present note we give necessary and sufficient conditions satisfied by every basic system of directions.

In order to formulate these necessary and sufficient conditions, we shall regard the coordinates $a_1, a_2, \dots, a_n$ of the vector $l \neq 0$ as homogeneous coordinates of the point $A = (a) = (a_1, a_2, \dots, a_n)$ of the $(n-1)$-dimensional projective space Π_{n-1}.

Theorem. *In order that the sequence of directions determined by the vectors $l_i = (a_i) \neq 0,\ i = 1, 2, \dots,$ be basic, it is necessary and sufficient that the sequence of points $A_i = (a_i)$ of the space Π_{n-1} not belong to any $(n-2)$-dimensional algebraic surface of this space.*

It follows from the theorem stated, for example, that the sequence of vectors $l_i = (a_i),\ i = 1, 2, \ldots$, determining a basic sequence of directions, must not be entirely contained in any hyperplane of the n-dimensional vector space.

1°. Proof of necessity. We shall show that if $(a_i) \in M$, $i = 1, 2, \ldots$, where M is some $(n-2)$-dimensional algebraic surface of the space Π_{n-1}, then in any domain D of n-dimensional Euclidean space there are functions continuous in this domain that are not limits of any uniformly convergent sequences of sums of the form (1).

Let

$$P(a_1, a_2, \ldots, a_n) = \sum_{m_1+m_2+\cdots+m_n=m} c_{m_1,m_2,\ldots,m_n} a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} = 0,$$

where $c_{m_1,m_2,\ldots,m_n}$ are certain constant numbers, $m_j \geq 0$, $j = 1, 2, \ldots, n$, is the equation of the surface M in homogeneous coordinates. Take pro-

an arbitrary point $x_0 \in D$ and choose $\delta > 0$ so that the ball $\overline{K}$,

$$\sum_{i=1}^{n} (x_i - x_{0i})^2 \leqslant \delta,$$

lies in the domain D. Consider the operator

$$O_v(u) = \iint\limits_{\overline{K}} u(x) L[v(x)]\, dx, \qquad dx = dx_1 \ldots dx_n,$$

where $L = P(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$, and $v(x)$ is an m-times continuously differentiable function in the ball $\overline{K}$, equal to zero together with all partial derivatives up to order $m-1$, inclusive, on the boundary of the ball $\overline{K}$; functions of this kind will be called admissible.

Every function $u(x)$ continuous in the ball $\overline{K}$ and of the form $u = \varphi(ax)$, $(a) \in M$, sends the operator $O_v(u)$ to zero for any admissible function $v(x)$. Indeed, substitute an arbitrary admissible function $v(x)$ and the function $u = \varphi(ax)$ into the operator $O_v(u)$, and, assuming that $a_1 \neq 0$ (which does not restrict generality), make under the integral sign and in the operator L the change of variables, putting

$$y_1 = a_1 x_1 + a_2 x_2 + \cdots + a_{nx} n, \qquad y_2 = x_2, \ldots, y_n = x_n. \tag{2}$$

We shall have

$$\iint\limits_{\overline{K}} \varphi(ax) L[v(x)]\, dx = \frac{1}{|a_1|} \iint\limits_{\overline{K}'} \varphi(y_1) \overline{L}[\overline{v}(y)]\, dy, \tag{3}$$

where $\overline{K}'$ is the image of the ball $\overline{K}$, and $\overline{L}$ is the result of transforming the operator L by means of formulas (2).

Since

$$\overline{L}[\varphi(y_1)] = L[\varphi(ax)] = P(a_1, a_2, \dots, a_n)\varphi^{(m)}(ax) = 0$$

for any m-times differentiable function φ, it is easy to see that the coefficient of $\partial^m/\partial y_1^m$ in the operator $\overline{L}$ is equal to zero. We note further that the function $\overline{v}(y) = v(x)$ vanishes together with all partial derivatives on the boundary of the domain $\overline{K}'$. In view of this, by repeated integration by parts of the right-hand side of relation (3) with respect to a variable different from y_1, we are convinced of the validity of the equality

$$O_v[\varphi(ax)] = 0$$

for any admissible function $v(x)$.

From what has been proved, and also from the additivity of the integral and the possibility of passing to the limit in the operator $O_v(u)$ for a fixed function $v(x)$, it follows that if the functions $\varphi_{ik}(a_i x)$, $i = 1, 2, \dots, k$; $k = 1, 2, \dots$, are continuous in the ball $\overline{K}$, then every function $f(x)$ that is the limit of a sequence, uniformly convergent in $\overline{K}$, of sums of the form (1) also sends the operator $O_v(u)$ to zero, whatever admissible function $v(x)$ may be.

To finish the proof it is necessary to show the existence of a function $u = u_0(x)$, continuous in the domain D, which does not send the operator $O_v(u)$ to zero for some admissible function $v(x)$.

If $c_{m_1^0, m_2^0, \dots, m_n^0}$ is a nonzero coefficient of the polynomial $P(a_1, a_2, \dots, a_n)$, then such a function will be, for example, the function

$$u_0(x) = x_1^{m_1^0} x_2^{m_2^0} \dots x_n^{m_n^0}.$$

For it and for the function

$$v_0(x) = \left[\delta^2 - \sum_{i=1}^{n} (x_i - x_{0i})^2\right]^m$$

we shall have

$$O_v(u_0) \neq 0,$$

and thus the necessity of the conditions of the theorem is proved.

2°. **Proof of sufficiency.** Let a sequence of directions, determined by vectors

$$l_i = (a_i) \neq 0, \qquad i = 1, 2, \dots,$$

be such that the corresponding sequence of points

$$A_i = (a_i) \tag{4}$$

of the space Π_{n-1} does not belong to any $(n-2)$-dimensional algebraic surface. The vectors $l_i,\ i = 1, 2, \dots,$ may clearly be assumed pairwise noncollinear.

For an arbitrary natural number m and any set of points $\{(a_{k_r})\} \subset \{(a_i)\}$, where k_r is a natural number, $r = 1, 2, \dots, N,\ N = C_{m+n}^{m},\ i = 1, 2, \dots,$ consider the identities

$$(a_{k_r 1} x_1 + a_{k_r 2} x_2 + \dots + a_{k_{rn}} x_n)^m =$$

$$= \sum_{m_1 + m_2 + \dots + m_n = m} \frac{m!}{m_1! m_2! \cdots m_n!} a_{k_r 1}^{m_1} a_{k_r 2}^{m_2} \cdots a_{k_{rn}}^{m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}, \tag{5}$$

$$m_j \geq 0, \qquad j = 1, 2, \dots, n,$$

which we take as a system of linear equations with respect to the quantities

$$\frac{m!}{m_1! m_2! \cdots m_n!} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}. \tag{6}$$

The determinant of this system is equal to

$$\Delta(a_{k_1}, \dots, a_{k_N}) = \left| a_{k_r 1}^{m_1} a_{k_r 2}^{m_2} \cdots a_{k_{rn}}^{m_n} \right|;$$

$$m_1 + m_2 + \dots + m_n = m, \qquad m_j \geq 0, \quad j = 1, 2, \dots, n.$$

We shall call a set of points $\{(a_{k_r})\},\ r = 1, \dots, N$, for which $\Delta(a_{k_1}, \dots, a_{k_N}) \neq 0$, a nondegenerate set of points of order m.

Let now $n = 2$. In this case, as the sequence $l_i,\ i = 1, 2, \dots,$ one may take any sequence of pairwise noncollinear vectors. It is also easy to see that for $n = 2$ every set of distinct points of the sequence (4) containing $m+1$ points is a nondegenerate set of order m, $m = 1, 2, \dots$. Therefore, putting $k_r = r,\ r = 1, 2, \dots, m+1$, and solving the system (5) with respect to the unknowns (6), we find that any product $x_1^{m_1} x_2^{m_2}$, $m_1 + m_2 = m,\ m_1, m_2 \geq 0$, and consequently any homogeneous polynomial of degree m in the variables x_1, x_2, is representable in the form of linear combinations of powers $(a_i x)^m$, $i = 1, 2, \dots, m+1$. Hence, and from the arbitrariness of m, it follows that every polynomial $P_k(x)$ of degree k is representable in the form

$$P_k(x) = \sum_{i=1}^{k+1} \varphi_i(a_i x), \tag{7}$$

where the functions $\varphi_i(t)$ are also polynomials of degree k in t.

Then, taking a sequence of polynomials $\{P_k(x)\}$, $k = 1, 2, \dots$, approximating the function $f(x)$ uniformly inside D, and using equality (7), we obtain the proof of sufficiency in the case $n = 2$.

For $n > 2$, an arbitrary set of points $\{(a_{k_r})\}$, $r = 1, 2, \dots, N$, of the sequence (4) need not be nondegenerate. We shall show, however, that for any natural m there exist nondegenerate sets of points belonging to the indicated sequence. To this end suppose the contrary and choose from the sequence (4) some particular ...

a system of mutually distinct points $\{(a^*_{k_r})\}$, $r = 1, 2, \dots, N$. Fix any $N-1$ of these points, and let the remaining point, for example $(a^*_{k_j})$, run through all points of the sequence (a_i), $i = 1, 2, \dots$. By assumption, each time $\Delta(a^*_{k_1}, \dots, a^*_{k_j}, \dots, a^*_{k_N}) = 0$. Since the original sequence does not belong to any $(n-2)$-dimensional algebraic surface, all minors of the j-th row of the determinant $\Delta(a^*_{k_1}, \dots, a^*_{k_N})$ are equal to zero. In view of the arbitrary choice of the set of points $\{(a_{k_r})\}$ and of the variable point $(a^*_{k_j})$ among the points of this set, it follows that, in general, any minor of order $N-1$ of the determinant $\Delta(a_{k_1}, \dots, a_{k_N})$, taken for an arbitrary set of points of the sequence (4), is equal to zero. Repeating the preceding argument for each minor of order $N-1$, we find that all minors of order $N-2$ of the determinant under consideration are also equal to zero for all possible sets of points of our sequence. Continuing in the same way, we arrive at the conclusion that all first-order minors of the determinant $\Delta(a_{k_1}, \dots, a_{k_N})$, formed from any points of the sequence (a_i), $i = 1, 2, \dots$, are identically equal to zero; but this contradicts the condition, since $l_i \neq 0$, $i = 1, 2, \dots$.

Thus, for the sequence (a_i), $i = 1, 2, \dots$, there exist nondegenerate sets of points of every order m. For each m we fix one such set, and substitute the coordinates of the points of this set into relations (5). If now $P_k(x)$ is an arbitrary polynomial of degree k, then, solving the resulting system (5) with respect to the unknowns (6) for $m = 1, 2, \dots, k$ and replacing in the polynomial $P_k(x)$ the powers $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, $m_1 + m_2 + \dots + m_n = m$, $m_j \geq 0$, by the results of the solution, we arrive at the equality

$$P_k(x) = \sum_{i=1}^{N_k} \varphi_i(a_i x),$$

where $\varphi_i(t)$, $i = 1, 2, \dots, N_k$, are certain polynomials in t of degree not exceeding k. The proof of the sufficiency of the conditions of the theorem is then obtained by referring to the possibility of uniformly approximating the function $f(x)$ inside the domain D by polynomials.

Using the preceding result, one may, for example, assert the following.

Let t_i, $i = 1, 2, \dots$, be a sequence consisting of an infinite set of mutually distinct real numbers converging to some number t_0; then the sequence of directions $(t_i, e^{t_i}, 1)$, $i = 1, 2, \dots$, in three-dimensional space is basic. The assertion follows from the fact that the endpoints of the vectors taken in the plane Π_2 do not lie on any algebraic curve.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.