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V. P. IL' IN

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Abstract

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V. P. IL' IN

ON THE APPROXIMATION OF FUNCTIONS FROM THE SPACES $\widetilde{W}_p^{(l)}(D)$ AND $W_p^{(l)}(D)$ BY CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

(Presented by Academician V. I. Smirnov on 28 XI 1960)

1. Let D be a finite or infinite domain of n-dimensional Euclidean space E_n, and let l be a positive (not necessarily integer) number.

By $\widetilde{W}_p^{(l)}(D)$ $(p \geq 1)$ we shall denote the set of all functions $f(X)$ $(X = (x_1, \dots, x_n))$, defined in D, possessing all generalized derivatives in the sense of S. L. Sobolev ([1]) of order $\bar{l} = [l]$ ($[l]$ is the integer part of l), satisfying the conditions:

$$1) \quad \|f\|_{L_p(D)} = \left[\int_D |f(X)|^p \, dX\right]^{1/p} < \infty;$$

$$2) \quad \|f\|_{L_p^{(l)}(D)} = \sum_{i_1,\dots,i_l=1}^{n} \left[\int_D \left|\frac{\partial^l f(X)}{\partial x_{i_1} \cdots \partial x_{i_l}}\right|^p dX\right]^{1/p} < \infty,$$

if l is an integer, or

$$2') \quad \|f\|_{L_p^{(l)}(D)} = \sum_{i_1,\dots,i_{\bar{l}}=1}^{n} \left[\int_D \left(\int_D \frac{\left|\frac{\partial^{\bar{l}} f(X)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}} - \frac{\partial^{\bar{l}} f(Y)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}}\right|^p}{|X - Y|^{n+(l-\bar{l})p}} \, dY\right) dX\right]^{1/p} < \infty,$$

if l is not an integer.

Put

$$\|f\|_{\widetilde{W}_p^{(l)}(D)} = \|f\|_{L_p(D)} + \|f\|_{L_p^{(l)}(D)}.$$

From the class of functions belonging to $\widetilde{W}_p^{(l)}(D)$, we single out the subclass of functions having all possible generalized derivatives of order up to $\bar{l}$ inclusive, belonging to $L_p(D)$. We shall denote this class of functions by $W_p^{(l)}(D)$.

Put

$$\|f\|_{W_p^{(l)}(D)} = \sum_{k=0}^{l} \|f\|_{L_p^{(k)}(D)},$$

if l is an integer, and

$$\|f\|_{W_p^{(l)}(D)} = \sum_{k=0}^{\bar{l}} \|f\|_{L_p^{(k)}(D)} + \|f\|_{L_p^{(l)}(D)}$$

if l is not an integer.

In the study of the spaces $\widetilde{W}_p^{(l)}(D)$ and $W_p^{(l)}(D)$, an important role is played by the problem of approximating a function $f \in \widetilde{W}_p^{(l)}(D)$ or $f \in W_p^{(l)}(D)$, in the norm $\widetilde{W}_p^{(l)}(D)$ or, respectively, $W_p^{(l)}(D)$, by means of a sequence of functions $\varphi_\nu(X)$ $(\nu = 1, 2, ...)$ belonging to $C^{(l)}(\overline{D})$, where $\overline{D} = D + \Gamma$, Γ is the boundary of D.

If $D = E_n$, then such an approximation with any degree of accuracy is carried out by means of averaging functions ([1]). If, however, D does not coincide with the whole space E_n, then, as is known, it is not always possible. For the case when the boundary of the domain D belongs to the class $C^{(l)}$, the possibility of such an approximation for functions $f \in W_p^{(l)}(D)$ was proved by V. M. Babich ([2]) (for integer l) and by L. N. Slobodetskii ([3]) (for noninteger l). For integer l, for the same class of functions, E. Gagliardo ([4]) obtained an analogous result under the assumption that D is a bounded domain whose boundary belongs to the class Lip 1. For functions belonging to $\widetilde{W}_p^{(l)}(D)$, the possibility of the above approximation was proved for the case when D is a bounded domain star-shaped with respect to some interior point D ([5]).

The results given below concern the same circle of questions.

II. Let D be a finite or infinite domain of the Euclidean space E_n. We introduce the following definitions:

1) We shall say that $D \in C(H, \sigma)$ if for each point $X \in D$ there exists an n-dimensional spherical sector with vertex at X, radius H, and solid angle σ, wholly contained in D.

2) We shall say that the domain D belongs to the class $C(H, \sigma, K, \lambda)$ and write $D \in C(H, \sigma, K, \lambda)$ if it satisfies the following condition: for any two points X and Y in D for which $|X - Y| \leq H$, there exist n-dimensional spherical sectors with solid angle σ and radius $\leq K|X-Y|$, with vertices at

X and Y, contained in D, and such that, if by G we denote the intersection of these sectors, then the inequality holds:

$$mG \geq \lambda |X - Y|^n,$$

where H, σ, K, λ are positive numbers fixed for the given domain.

It is evident that if $D \in C(H, \sigma, K, \lambda)$, then $D \in C(H/2, \sigma)$.

3) By D_δ we shall denote the domain consisting of the points of D whose distance from the boundary of D is greater than δ.

4) We shall say that the domain D has property $A(N, \varkappa)$, and write $D \in A(N, \varkappa)$, if there exist two finite systems of n-dimensional domains $S_1, \dots, S_N$ and $S'_1, \dots, S'_N$, each of which forms a covering of D, satisfying the following conditions: a) $\overline{S}_i \subset S''_i$ $(i = 1, \dots, N)$, and if $X \in S_i$, $Y \in \overline{S''_i}$, then $|X - Y| \geq \varkappa > 0$; b) the sets $D_i = D \cdot S_i$, $D'_i = D \cdot (S'_i)_{\varkappa/2} = D \cdot S'_i$, $D''_i = D \cdot S''_i$ $(i = 1, \dots, N)$, as well as the sets $D'_i \cdot D'_k$ $(i, k = 1, \dots, N)$, if they are nonempty, are finitely connected; c) for each set $D'_i = D \cdot S'_i$ $(i = 1, \dots, N)$ there exists a vector Q_i such that the translation of D'_i by the vector tQ_i, for arbitrary t, $0 < t \leq 1$, carries D'_i into a domain Ω''_{it} interior with respect to D, i.e., such that

$$\rho(\Omega''_{it}, E_n - D) > 0 \quad (i = 1, \dots, N).$$

We note that if some domain S'_i is a strictly interior subdomain of the domain D, then the corresponding vector Q_i may be taken to be zero.

We also note that if D is a bounded domain whose boundary belongs to the class Lip 1, then there exist positive numbers $H, \sigma, K, \lambda, N, \varkappa$ such that $D \in C(H, \sigma, k, \lambda)$ and $D \in A(N, \varkappa)$.

III. In all the theorems given below, $\varphi_\nu(X)$ will denote functions having continuous derivatives of every order in the whole space E_n.

Theorem 1*. If $f \in W_p^{(l)}(D)$, $p \geqslant 1$, l is an integer, $D \in A(N, \varkappa)$, then there exists a sequence of functions $\varphi_\nu(X)$ $(\nu = 1, \dots)$ such that

$$\lim_{\nu \to \infty} \|f - \varphi_\nu\|_{W_p^{(l)}(D)} = 0.$$

Lemma. Let $f(X) \in \widetilde{W}_p^{(l)}(D)$ and have continuous derivatives up to order $\overline{l} = [l]$ in D, $D \in C(H, \sigma)$. Then for any integer s, $0 \leqslant s \leqslant \overline{l}$, the inequality

$$\|f\|_{L_p^{(s)}(D)} \leqslant C_1 \|f\|_{L_p(D)}^{1-s/l} \|f\|_{\widetilde{W}_p^{(l)}(D)}^{s/l}.$$

holds.

If l is not an integer, $D \in C(H, \sigma, K, \lambda)$, then the inequality

$$\|f\|_{L_p^{(s+l-\bar{l})}(D)} \leqslant C_2 \|f\|_{L_p(D)}^{\frac{\bar{l}-s}{l}} \|f\|_{\widetilde{W}_p^{(l)}(D)}^{1-\frac{\bar{l}-s}{l}}.$$

also holds.

The constants C_1 and C_2 do not depend on f.

With the help of this lemma the following theorems are proved.

Theorem 2. If $f \in \widetilde{W}_p^{(l)}(D)$, $p \geqslant 1$, l is an integer, $D \in A(N, \varkappa)$ and $D \in C(H, \sigma)$, then:

1) $f \in W_p^{(l)}(D)$;
2) there exists a sequence of functions $\varphi_\nu(X)$ $(\nu = 1, \ldots)$ such that

$$\lim_{\nu\to\infty} \|f - \varphi_\nu\|_{W_p^{(l)}(D)} = 0.$$

Theorem 3. If $f \in \widetilde{W}_p^{(l)}(D)$, $p \geqslant 1$, l is not an integer, $D \in A(N, \varkappa)$, $D \in C(H, \sigma, K, \lambda)$, then:

1) $f \in W_p^{(l)}(D)$;
2) there exists a sequence of functions $\varphi_\nu(X)$ $(\nu = 1, \ldots)$ such that

$$\lim_{\nu\to\infty} \|f - \varphi_\nu\|_{W_p^{(l)}(D)} = 0.$$

Remark. Let $f(X)$ and the domain D satisfy the conditions of Theorem 2 if l is an integer, or the conditions of Theorem 3 if l is not an integer. Suppose, in addition, that for any integer m, $0 \leqslant m \leqslant n$, and all derivatives of order $\bar{l} = [l]$ the inequalities

$$\sup_{D_m} \left[\int_{(|D_m|_{n-m}^d)} \cdots \int |D^l f(X)|^p \, dX \right]^{1/p} \leqslant M d^{\alpha_m}, \tag{1}$$

hold if l is an integer, or

$$\sup_{D_m} \left[\int_{(|D_m|_{n-m}^d)} \cdots \int \left(\int_{(D)} \cdots \int \frac{|D^{\bar{l}} f(X) - D^{\bar{l}} f(Y)|^p}{|X - Y|^{n+(l-\bar{l})p}} \, dY \right) dX \right]^{1/p} \leqslant M d^{\alpha_m}, \tag{2}$$

if l is not an integer, where $M > 0$, α_m $(m = 0, 1, \ldots, n)$ are constants, with

$$\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_n = 0, \qquad \alpha_m \leqslant \frac{n-m}{p},$$

and $|D_m|^d_{n-m}$ is the set of points of the domain D at distance no greater than d from some section D_m of the domain D by the hyperplane $x_{m+1} = \text{const}, \dots, x_n = \text{const}$.

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* If D is a bounded domain, then Theorem 1 is implicitly contained in the results of E. Gagliardo (4).

It can be shown that inequalities (1) and (2) will also hold for the functions $\varphi_\nu(X)$ constructed respectively in Theorems 2 and 3.

This remark makes it possible to extend the results of works [6,7] also to functions $f(X) \in \widetilde{W}_p^{(l)}(D)$, if D satisfies the conditions of Theorem 2 or 3.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.