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MATHEMATICS

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1961

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Abstract

Full Text

MATHEMATICS

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ON THE PROBLEM OF GLUING TWO HALF-DISKS

(Presented by Academician M. A. Lavrent'ev on 30 V 1961)

In the present note a class of Riemann surfaces is indicated, the determination of whose type is directly reduced to the determination of the type of the gluing of two half-disks for a given gluing function from a sufficiently broad class.

1. Let $\bar{x} = \varphi(x)$ be a topological correspondence between the points of the boundary diameters of the half-disks $D_1(\text{Im } z \leq 0, |z| < \rho)$ and $D_2(\text{Im } z \geq 0, |z| < \rho)$, $\varphi(\pm\rho) = \pm\rho$, $\rho \leq \infty$, such that there exists a pair of analytic functions $\hat{z} = f_1(z)$ and $\hat{z} = f_2(z)$, regular and univalent in D_1 and D_2 , which transform the latter into domains filling the disk $|\hat{z}| < R \leq \infty$, while $f_1[\varphi(x)] = f_2(x)$ for all $x \in (-\rho, \rho)$. For convenience, instead of D_1, D_2 we consider the half-strips B_r , $r = 1, 2$: $(-1)^r \text{Im } z \geq 0, |\text{Re } z| < \rho$, with the same correspondence between the points of adjoining bases. The passage from the problem of gluing D_r to the problem of gluing B_r with gluing function $\varphi(x)$ is ensured by an elementary q -quasiconformal transformation. The half-strips B_r with gluing function $\varphi(x)$ are understood as a connected univalent domain and are denoted by Φ ($\varphi, \rho \leq \infty$). If F is an arbitrary simply connected Riemann surface, and $z = f(w)$ is a function mapping F conformally onto the disk $|z| < R \leq \infty$, then the distribution of the characteristics $p(w), \theta(w)$ on F preserves the type in the case when there exists a quasiconformal mapping with characteristics $p(z), \theta(z)$ of the disk $|z| < R$ onto the disk $|\hat{z}| < R$ ⁽¹⁾.

Let $\{a_{rk}\}$, $r = 1, 2$, $a_{rk} < a_{rk+1}$, $k = 0, \pm 1, \pm 2, \dots$, be arbitrary sequences of real numbers, $\lim a_{rk} = \pm\infty$ as $k \rightarrow \pm\infty$ and $r = 1, 2$. In each half-strip $S_{rk} : u \in (a_{rk-1}, a_{rk})$, $(-1)^r v > 0$, on a copy of the w -plane, $w = u + iv$, on the sheet H make cuts along the segments

$$w = u + i(-1)^r \frac{2n-1}{m_{rk}}, \quad n = 1, 2, \dots, \quad a_{rk-1} + m_{rk}^{-1} <$$

$$< u < a_{rk} - m_{rk}^{-1},$$

where $\{m_{rk}\}$ are two sequences of real numbers,

$$m_{rk} > 2(a_{rk} - a_{rk-1})^{-1} = 2\Delta_{rk}^{-1}$$

for all k and r . To each cut we glue a copy of the w -plane—the sheet H_{nk}^r with an analogous cut. The open simply connected Riemann surface constructed in this way has, over each disk $|w| < R$, only algebraic branch points of the first order. Denote by \mathfrak{A} the class of surfaces of this kind obtained under all possible choices of the sequences $\{a_{rk}\}$ and $\{m_{rk}\}$. The rectangle

$$R_{nk}^r : u \in (a_{rk-1}, a_{rk}), \quad v \in \left((-1)^r \frac{2n}{m_{rk}}, (-1)^r \frac{2(n+1)}{m_{rk}} \right)$$

with the sheet H_{nk}^r attached to it is called an element of the surface $F \in \mathfrak{A}$ and is denoted by $G_{nk}^r(\Delta_{rk}, m_{rk})$.

Definition. A surface $F \in \mathfrak{A}$ is called an **equivalent domain** $\Phi(\varphi, \rho \leq \infty)$ if, when a cut is made along the real axis of the sheet H , dividing F into two simply connected parts F_1 and F_2 , one can define a pair of quasiconformal mappings $g_1(w)$ and $g_2(w)$, with characteristics preserving the type of F , such that $g_1(F_1) = B_1$, $g_2(F_2) = B_2$,

$g_1(\gamma) = g_2(\gamma) = \pm\rho$, where γ is a boundary element F , and for all x , $\varphi(x) = g_2[g_1^{-1}(x)]$.

2. We further assume that the quantity $\frac{1}{2}\Delta_{rk} - m_{rk}^{-1}$ is of order not less than m_{rk}^{-1} as $k \rightarrow \pm\infty$. Let β_{nk}^r be the center of the rectangle R_{nk}^r , and let $w^* = L_{nk}^r(w)$, $w^* = u^* + iv^*$, be the translation transformation, $L_{nk}^r(\beta_{nk}^r) = 0$. Put

$$\hat{G}_{nk}^r = L_{nk}^r(G_{nk}^r(\Delta_{rk}, m_{rk})).$$

Lemma. One can construct a q -quasiconformal mapping $\omega = g_{rk}(w^*)$ of the element \hat{G}_{nk}^r onto a univalent rectangle

$$|\operatorname{Im} \omega| < m_{rk}^{-1}, \quad |\operatorname{Re} \omega| < (\pi m_{rk})^{-1} \ln m_{rk} \Delta_{rk} (1 + \varepsilon_{rk}),$$

where $\varepsilon_{rk} \rightarrow 0$ as $\Delta_{rk} m_{rk} \rightarrow \infty$, such that $|g'_{rk}(w^*)| \equiv 1$ on the lateral sides \hat{R}_{nk}^r , and on the horizontal bases of \hat{R}_{nk}^r , $\hat{R}_{nk}^r = L_{nk}^r(R_{nk}^r)$,

$$g_{rk} \left(u^* \pm i \frac{1}{m_{rk}} \right) = \frac{1}{\pi m_{rk}} \ln \frac{l_{rk} + u^* \left[1 - \frac{2}{m_{rk} \Delta_{rk}} (1 + 2\lambda_{rk}) \right]}{l_{rk} - u^* \left[1 - \frac{2}{m_{rk} \Delta_{rk}} (1 + 2\lambda_{rk}) \right]} \pm i \frac{1}{m_{rk}}.$$

Here

$$l_{rk} = \frac{1}{2} \Delta_{rk} - \frac{1}{m_{rk}} \left(\frac{3}{2} \lambda_{rk} + 1 \right), \quad \lambda_{rk} < K = \text{const.}$$

Theorem 1. Let $\varphi(x)$, $\varphi'(x) > 0$, be a continuously differentiable gluing function of the half-strips B_1 and B_2 , admitting conformal gluing. Then in the class \mathfrak{A} there exists a surface equivalent to the domain $\Phi(\varphi, \rho \leq \infty)$.

Proof. On the interval $(-\rho, \rho)$ consider a continuous piecewise-linear function $\bar{x} = \psi(x)$, $\psi'(x) > 0$, $\psi(\pm\rho) = \pm\rho$, which for all x satisfies the condition

$$\max\left(\frac{\varphi'(x)}{\psi'(x)}, \frac{\psi'(x)}{\varphi'(x)}\right) < K, \quad (1)$$

where $K = \text{const}$. Let $\{x_k\}$, $x_k < x_{k+1}$, $k = 0, \pm 1, \pm 2, \dots$, $\lim_{k \rightarrow \pm\infty} x_k = \pm\rho$, be the set of discontinuity points of $\psi'(x)$. The choice of such a function under the hypotheses of the theorem is possible. The deformation $\xi = \eta(z) = \varphi[\psi^{-1}(x)] + iy$ for $\text{Im } z > 0$ and $\eta(z) \equiv z$ for $\text{Im } z < 0$, under condition (1), provides a q -quasiconformal transition from $\Phi(\psi, \rho \leq \infty)$ to $\Phi(\varphi, \rho \leq \infty)$. Let $\{\alpha_k\}$, $\alpha_k < \alpha_{k+1}$, $k = 0, \pm 1, \pm 2, \dots$, $\lim_{k \rightarrow \pm\infty} \alpha_k = \pm\infty$, be a sequence of real numbers, whose choice we shall specify below. For each k define a collection of elements

$$\{G_{n_{j_{rk}}}^r = G_{n_{j_{rk}}}^r(\Delta_{rj_{rk}}, m_{rj_{rk}})\}, \quad j_{rk} = n_{rk}, n_{rk} + 1, \dots, n_{rk+1} - 1, \\ \Delta_{rj_{rk}} = \Delta_{rj_{rk+1}}, \quad m_{rj_{rk}} = m_{rj_{rk+1}} \quad \text{for } n_{rk} \leq j_{rk} \leq n_{rk+1} - 2,$$

connected by translation transformations

$$G_{ni'}^r = T_{rk}^{(i')}(G_{n_{n_{rk}}}^r), \quad G_{i''j_{rk}}^r = \hat{T}_{rk}^{(i'')}(G_{1j_{rk}}^r),$$

where $T_{rk}^{(i')}$ and $\hat{T}_{rk}^{(i'')}$ are superpositions of i' and i'' transformations, respectively,

$$T_{rk} = w + \Delta_{rn_{rk}}, \quad \hat{T}_{rk} = w + (-1)^r \frac{2i}{m_{rn_{rk}}}.$$

The corresponding rectangles $\{R_{n_{j_{rk}}}^r\}$ fill the half-strips

$$S_{rk} : \quad (-1)^r v > 0, \quad \alpha_k < u < \alpha_{k+1},$$

univalently, and $R_{1n_{rk}}^r$ is the rectangle

$$v \in \left(0, (-1)^r \frac{2i}{m_{rn_{rk}}}\right), \quad u \in (\alpha_k, \alpha_k + \Delta_{rn_{rk}}).$$

Let

$$F = \bigcup_{k,r} F_{rk}, \quad F_{rk} = \bigcup_{n, j_{rk}} \bar{G}_{n_{j_{rk}}}^r.$$

Obviously, $F \in \mathfrak{A}$. Define on the interval $(0, \infty)$ a continuous function $h(\xi) > 0$, subjecting it to the conditions

$$h(\xi) \rightarrow \infty, \quad \frac{1}{\xi} h(\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow \infty.$$

Put

$$\Delta_{rj_{rk}} = \frac{1}{h(m_{rk})},$$

where $m_{rk} = m_{rn_{rk}}$, for all k and r . In this case each element belonging to F_{rk} is transformed q -quasiconformally onto a univalent ...

rectangle with base length

$$d_{rk} = \frac{2}{\pi m_{rk}} \ln \frac{m_{rk}}{h(m_{rk})} (1 + \varepsilon_{rk})$$

and height $2m_{rk}^{-1}$. If m_{rk} is now determined from the equalities

$$\frac{2}{\pi} \frac{h(m_{rk})}{m_{rk}} \ln \frac{m_{rk}}{h(m_{rk})} (1 + \varepsilon_{rk}) = \frac{\delta_{rk+1} - \delta_{rk}}{\alpha_{k+1} - \alpha_k}, \quad (2)$$

where $\delta_{1k} = x_k$ and $\delta_{2k} = \psi(x_k)$, then the surface F_{rk} is mapped q -quasiconformally, under a suitable normalization of the mappings of the lemma, onto half-strips in the plane z_1 ,

$$\hat{S}_{rk} : \quad (-1)^r \operatorname{Im} z_1 > 0, \quad \delta_{rk} < \operatorname{Re} z_1 < \delta_{rk+1}.$$

Denote these mappings by $z_1 = v_{rk}(w)$ and note that $v'_{rk}(w) = g'_{rk}(w)$, where $g_{rk}(w)$ is the mapping of the lemma. In \hat{S}_{rk} consider an additional deformation $z = v_{rk}^*(z_1)$, transforming \hat{S}_{rk} into itself, which outside the semicircles

$$P_{rj'_{rk}} : \quad |z_1 - \delta_{rk} - d_{rk}(j'_{rk} + 1/2)| < \frac{1}{2}d_{rk}, \quad (-1)^r \operatorname{Im} z_1 > 0,$$

$j'_{rk} = j_{rk} - n_{rk}$, coincides with the identity mapping, and inside the latter

$$v_{rk}^*(z_1) = \mu^{-1}\{\chi[\mu(z_1)]\}.$$

Here (we omit the indices) $z_2 = \mu(z_1)$ is a q -quasiconformal mapping of the semicircle $P_{rj'_{rk}}$ onto the half-plane $(-1)^r \operatorname{Im} z_2 > 0$,

$$\mu(\delta_{rk} + d_{rk}(j'_{rk} + 1)) = 1, \quad \mu(\delta_{rk} + d_{rk}j'_{rk}) = 0,$$

$$\mu(\delta_{rk} + d_{rk}(j'_{rk} + 1/2)) + (-1)^r i d_{rk}/2 = (-1)^r i; \quad |\mu'(\operatorname{Re} z_1)| = 1/d_{rk},$$

and $\chi(z_2)$ is a quasiconformal mapping of the form

$$\chi(z_2) = L[\hat{v}_{rk}^{-1}(\operatorname{Re} z_2)] + i \operatorname{Im} z_2,$$

where $\hat{v}_{rk}(w) = \mu[v_{rk}(w)]$, and $L(w)$ is an integral linear function,

$$L[\widehat{v}_{rk}^{-1}((0, 1))] = (0, 1), \quad L'(w) = 1/\Delta_{rj_{rk}},$$

for points of the half-strip

$$0 < \operatorname{Re} z_2 < 1, \quad (-1)^r \operatorname{Im} z_2 > 0,$$

and $\chi(z_2) \equiv z_2$ for the remaining points of the half-plane $(-1)^r \operatorname{Im} z_2 > 0$. The resulting mapping $z = f_{rk}(w)$ maps F_{rk} quasiconformally onto the half-strips \widehat{S}_{rk} , with constant boundary stretching, and at the points of the boundary half-lines of F_{rk}

$$|f'_{rk}(w)| \equiv 1.$$

Thus the function $z = f_r(w)$, $f_r(w) = f_{rk}(w)$ for $w \in F_{rk}$, maps the selected surface quasiconformally onto $\Phi(\psi, \rho \leq \infty)$. Since $v'_{rk} = g'_{rk}$, we have

$$\frac{1}{K} \frac{1}{m_{rk} \Delta_{rj_{rk}}} < v'_{rk} < K,$$

where K is a constant independent of k and r , for all $u \in (\alpha_k, \alpha_{k+1})$. Therefore, for $z_2 \in \mu \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right)$,

$$p_\chi(z_2) < \max \left(\frac{L'}{\mu' v'_{rk}}, \frac{\mu' v'_{rk}}{L'} \right) < K \frac{m_{rk}}{h(m_{rk}) \ln(m_{rk}/h(m_{rk}))}. \quad (3)$$

By construction, $p_{f_r}(z)$ at the points of the set $\bigcup_{k, j'_{rk}} P_{rj'_{rk}}$ does not exceed the right-hand side of (3), possibly with another constant factor, and outside it is uniformly bounded. Under the conditions of the theorem there exists a pair of functions regular and univalent in the half-strips $(-1)^r \operatorname{Im} \zeta \geq 0$, $|\operatorname{Re} \zeta| < \rho \leq \infty$,

$$\tau = t_r(\zeta),$$

which map $\Phi(\varphi, \rho \leq \infty)$ onto the disk $|\tau| < R \leq \infty$. Put

$$t(\zeta) = t_r(\zeta) \quad \text{for } \zeta \in B_r, \quad \zeta = \eta(w) = \eta[f_r(w)].$$

Let $\tau_1 = t_1(\tau)$ be a quasiconformal mapping of the annulus $R_0 < |\tau| < R$ onto the annulus $R_0 < |\tau_1| < R_1$, with characteristic

$$p = p_{\tilde{\eta}}(\tau) \quad \text{for } \tau \in t \left[\eta \left(\bigcup_{k, j'_{rk}} P_{rj'_{rk}} \right) \right] \cap (R_0 < |\tau| < R) = E_r,$$

and $p = 1$ on the complementary set. By the Belinskii-Helly theorem (2,3),

$$\left| \ln \frac{R}{R_1} \right| \leq \iint_{R_0 < |\tau| < R} \frac{p-1}{|\tau|^2} d\sigma_\tau < \sum_{r=1,2} \iint_{E_r} \frac{p_{\tilde{\eta}}(\tau)}{|\tau|^2} d\sigma_\tau < K \sum_{r=1,2} \iint_{E_r} \frac{p_{f_r}(\tau)}{|\tau|^2} d\sigma_\tau. \quad (4)$$

Here $d\sigma_\tau$ is the area element in the τ -plane. To estimate the integral on the right-

of the right-hand part of (4), we pass to the ζ -plane. We have

$$\left| \ln \frac{R}{R_1} \right| < K \sum_{r=1,2} \iint_{t^{-1}(E_r)} p_{f_r}(\zeta) \left| \frac{d \ln t(\zeta)}{d\zeta} \right|^2 d\sigma_\zeta < K \sum_{r, k \geq k_0} \text{mes } \eta \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right) p_{rk} M_{rk}, \quad (5)$$

where $p_{rk} = \max p_{f_r}(\zeta)$, $M_{rk} = \max |d \ln t(\zeta)/d\zeta|^2$ for $\zeta \in \eta \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right)$, and the summation is carried out over those sets $\eta \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right)$ for which $\eta \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right) \cap t^{-1}(E_r) \neq \emptyset$. Since, by the definition of the mapping $\zeta = \eta(z)$, we have

$$\text{mes } \eta \left(\bigcup_{j'_{rk}} P_{rj'_{rk}} \right) \leq \max_{x \in \delta_{rk}} \left| \frac{\partial \eta}{\partial x} \right| \text{mes } \bigcup_{j'_{rk}} P_{rj'_{rk}} < K \frac{\delta_{rk+1} - \delta_{rk}}{m_{rk}} \ln \frac{m_{rk}}{h(m_{rk})}, \quad (6)$$

then, assuming $\delta_{rk+1} - \delta_{rk} < O(1)$, from (3), (5), and (6) we finally obtain

$$\left| \ln \frac{R}{R_1} \right| < K \sum_{r, k \geq k_0} \frac{M_{rk}}{h(m_{rk})}. \quad (7)$$

After $\psi(x)$ has been chosen, the multiplier $\delta_{rk+1} - \delta_{rk}$ in the right-hand side of (2) is fixed, and moreover

$$\frac{h(\xi)}{\xi} \ln \frac{\xi}{h(\xi)} \rightarrow 0 \quad \text{as } \xi \rightarrow \infty.$$

Choose $\alpha_{k+1} - \alpha_k$ so that, for m_{rk} , determined from (2), the inequality

$$h(m_{rk}) > \max_{r=1,2} M_{rk} \lambda_k$$

holds, where

$$\sum_k \frac{1}{\lambda_k} < \infty.$$

Then it follows from (7) that, for these $\{\alpha_k\}$ and the chosen $\psi(x)$ and $h(\xi)$, R and R_1 are simultaneously finite or infinite. We now define a quasiconformal mapping of the annulus

$$R_0 < |\tau_1| < R_1$$

onto the annulus

$$R_0 < |\tau_2| < R_2$$

with characteristic $\hat{p} = p_{f_r}(\tau_1)$ outside the set $t_1(E_r)$, and $\hat{p} = 1$ on the complement. This mapping is q -quasiconformal; therefore

$$\frac{1}{K}R_1 < R_2 < KR_1,$$

where $K = \text{const}$. Thus the resulting quasiconformal mapping of the annulus $R_0 < |\tau| < R$ with characteristic $p_{f_r}(\tau)$ changes its modulus by no more than $K = \text{const}$ times. The theorem is proved.

3. Denote by $\mathfrak{A}[\Phi(\varphi, \rho < \infty)]$ the class of all surfaces from \mathfrak{A} equivalent to $\Phi(\varphi, \rho < \infty)$. By what has been proved, it is nonempty and contains, for example, surfaces equivalent to $\Phi(\varphi, \rho < \infty)$ obtained for different choices of the approximating function $\psi(x)$, the function $h(\xi)$, and the sequence $\{\alpha_k\}$. The validity of the following theorem is obvious.

Theorem 2. *All surfaces of the class $\mathfrak{A}[\Phi(\varphi, \rho < \infty)]$ are of hyperbolic type.*

Various sufficient type criteria for $\Phi(\varphi, \rho = \infty)$, each time applicable to all surfaces from $\mathfrak{A}[\Phi(\varphi, \rho = \infty)]$, were obtained in (4–9).

In conclusion I express my gratitude to A. A. Goldberg for a number of valuable comments.

Received

24 V 1961

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