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Abstract

Full Text

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On Higher Approximations in Averaging

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§ 1. Statement of the problem

In (^{3}) an averaging method was developed for systems

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \qquad \dot{y} = Y(x, y, t, \varepsilon) \tag{1}$$

(x, X are n-dimensional, y, Y are m-dimensional vectors, $\varepsilon > 0$ is a small parameter). For $\varepsilon = 0$, (1) passes into the degenerate system

$$\dot{y} = Y_0(x, y, t) \equiv Y(x, y, t, 0), \qquad x = \text{const}, \tag{2}$$

whose general solution is assumed known:

$$y = \varphi(x, y_0, t_0, t) \qquad (\varphi(x, y_0, t_0, t_0) \equiv y_0). \tag{3}$$

It is assumed that the right-hand sides of (1) and certain other functions have mean values along the trajectories (3): the mean value of the function $F(x, y_0, t_0, y, t)$ is taken to be

$$\bar{F}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(F\,|_{y=\varphi(x,y_0,t_0,t)}\right) dt$$

(here and below it is assumed everywhere that $\bar{F}(x)$ is continuous and satisfies the Lipschitz condition), and the limit exists uniformly with respect to x, y_0, t_0 in the domain under consideration and does not depend on y_0, t_0. (It is shown in (^{3}) that the assumption of independence of y_0, t_0 does not restrict generality.) In (^{3}) it was proved that, under certain conditions, the solution x of system (1) is approximated with error $\alpha(\varepsilon)$ ($\alpha(\varepsilon)$ here and below denotes quantities for which $\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 0$) on the interval $t \sim 1/\varepsilon$ by the solution of the averaged system of the first approximation

$$\dot{x} = \varepsilon \bar{X}_1(x)$$

$$(X_1(x, y, t) \equiv X(x, y, t, 0)).$$

In the present article the averaged system of the second approximation is considered,

$$\dot{\bar{x}} = \varepsilon \bar{X}_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}), \qquad \dot{\bar{y}} = Y_0(\bar{x}, \bar{y}, t) + \varepsilon B_1(\bar{x}), \tag{4}$$

derived in (³) (A_2, B_1 are indicated below). It is proved that, under certain conditions, the solutions of (4) approximate the solutions of (1) with error $\varepsilon\alpha(\varepsilon)$ for x and $\alpha(\varepsilon)$ for y on the interval $t \sim 1/\varepsilon$.

§ 2. Theorems on higher approximations

The domain of definition of (1): $0 \le \varepsilon \le \varepsilon_0$; $x, y, t \in G$, where G is an open domain.

Let:

1) X, Y be bounded, continuous, and have continuous uniformly bounded derivatives with respect to x, y, t, while with respect to ε they have derivatives up to the second order inclusive, with $X''_{\varepsilon^2}, Y''_{\varepsilon^2}$ uniformly bounded.

2) Through each point of the domain G there passes a unique integral curve (3) of system (2), lying in G for $t_0 \leqslant t < \infty$, continued for $t \leqslant t_0$ to the boundary of G or to $t \to -\infty$. The function (3) is continuous, and with respect to y_0, t_0 has continuous bounded derivatives up to the second order inclusive. $0 < c_1 \leqslant |\operatorname{Det} D| \leqslant c_2 < \infty$, where $D \equiv \partial\varphi/\partial y_0$ (here and below $\partial\varphi/\partial y_0$, $\partial Y/\partial y$, etc. denote the matrices $\|\partial\varphi_i/\partial y_{0k}\|$, $\|\partial Y_i/\partial y_k\|$, etc.).

3) In G there lies an $(n+m)$-dimensional manifold M, given parametrically: $x = a(\lambda)$, $y = b(\lambda)$, $t = c(\lambda)$ ($\lambda = \{\lambda_1, \dots, \lambda_{n+m}\} \in \Lambda$, Λ an open domain); a, b, c are continuous and have continuous bounded derivatives;

$$\sum_{i=1}^{n+m+1} A_i^2 \geqslant \text{const} > 0,$$

where A_i are the minors of order $(n+m)$ of the matrix $\|\partial a/\partial\lambda,\ \partial b/\partial\lambda,\ \partial c/\partial\lambda\|$. The absolute values of the angles of intersection of (3) with M are bounded below by a positive constant. Every curve (3) from G intersects M once.

4) There exists the mean value $\overline{X}_1$ of the function X_1 (here and below mean values are understood in the sense of the definition of § 1). The function $S \equiv X_1 - \overline{X}_1$ is uniformly bounded.

5) For $0 < \varepsilon \leqslant \varepsilon_0$ there exist open bounded subdomains $G_0(\varepsilon) \subseteq G$, containing the fixed initial point x_0, y_0, t_0 together with some ρ-neighborhood ($\rho = \text{const} > 0$). The time of passage of curve (3) from any point of G_0 to the intersection with M does not exceed, in absolute value, K/ε ($K = \text{const} > 0$). For $0 < \varepsilon \leqslant \varepsilon_0$ there exist open subdomains $G_1(\varepsilon) \subset G_0$, containing x_0, y_0, t_0; the distances from the points of G_1 to the boundary of G_0 are bounded below by a positive constant.

Under conditions 1)–5), in G there exists a continuously differentiable solution of the equation

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial y} Y_0 = S, \quad u_1(x,y,t)\big|_{x,y,t\in M} = 0,$$

and u_1 is easily constructed from the characteristics (3) and is therefore regarded as known.

6) Let u_1 be bounded in G and have bounded continuous derivatives with respect to x, y, t.
7) $\dfrac{\partial X_1}{\partial x} u_1$ has a mean value.
8) For $\partial u_1/\partial x,\ \partial u_1/\partial y$ there exist mean values.
9) $X_2 \equiv X'_\varepsilon\big|_{\varepsilon=0}$ also has mean values.
10) There exists the mean value $H(x)$ of the matrix D^{-1}, $0 < c_1 \leqslant |\operatorname{Det} H| \leqslant c_2 < \infty$.
11) There exists the mean value $R(x)$ of the function $D^{-1}\left(Y_1 + \dfrac{\partial Y_0}{\partial x} u_1\right)$ ($Y_1 \equiv Y'_\varepsilon\big|_{\varepsilon=0}$).
12) There exists the mean value of the expression

$$\frac{\partial X_1}{\partial y} D \int_{t_0}^{t} \left[D^{-1}\left(Y_1 + \frac{\partial Y_0}{\partial x} u_1 - B_1\right)\right]\bigg|_{y=\varphi(x,y_0,t_0,t)} dt,$$

where $B_1(x) \equiv H^{-1}R$ (B_1 enters system (4)).

From conditions 1)–12) it follows that in G there is defined a continuously differentiable solution of the equation

$$\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial y} Y_0 - \frac{\partial Y_0}{\partial y} v_1 = Y_1 + \frac{\partial Y_0}{\partial x} u_1 - B_1, \quad v_1\big|_{x,y,t\in M} = 0,$$

and moreover $\dfrac{\partial X_1}{\partial y} v_1$ has a mean value,—this is easily verified, since v_1 can be constructed from the characteristics, as can u_1. From what has been said it follows that the function

$$P \equiv \frac{\partial X_1}{\partial x} u_1 + \frac{\partial X_1}{\partial y} v_1 - \frac{\partial u_1}{\partial x} \overline{X}_1 - \frac{\partial u_1}{\partial y} B_1 + X_2$$

has a mean value $\equiv A_2(x)$ (A_2 enters system (4)).

Let:

13) The function $P - A_2$ is uniformly bounded.

14) The expressions

$$\frac{\partial}{\partial y_0}\left(\int_{t_0}^{t_0+T} (P - A_2)\, dt\right), \qquad \frac{\partial}{\partial t_0}\left(\int_{t_0}^{t_0+T} (P - A_2)\, dt\right),$$

$$\frac{\partial}{\partial y_0}\left(\int_{t_0}^{t_0+T} D^{-1}\left(Y_1 + \frac{\partial Y_0}{\partial x} u_1 - B_1\right) dt\right), \qquad \frac{\partial}{\partial t_0}\left(\int_{t_0}^{t_0+T} D^{-1}\left(Y_1 + \frac{\partial Y_0}{\partial x} u_1 - B_1\right) dt\right)$$

(the integrals are taken along (3)) are uniformly bounded for $0 \le T < \infty$.

15) System (4) possesses "stability" : for every $K > 0$ there exist $c_1 > 0$, $c_2 > 0$, $\bar{\varepsilon} > 0$ ($\bar{\varepsilon} \le \varepsilon_0$) such that if on some interval $[t_0, \bar{t}(\varepsilon)] \subseteq [t_0, K/\varepsilon]$, for $0 < \varepsilon \le \bar{\varepsilon}$, the solutions of (4) and of the equation

$$z = Y_0(x, z, t) + \varepsilon B_1(x) + \varepsilon \varphi(t, \varepsilon)$$

exist for the initial values x_0, y_0, t_0 and z_0, t_0 ($\|z_0 - y_0\| \le c_1$, x, y is the solution of (4) with initial point x_0, y_0, t_0), and $\varphi(t, \varepsilon)$ is an arbitrary continuous function such that

$$\sup_{t_0 \le t \le \bar{t}} |\varphi(t, \varepsilon)| \le c_1,$$

then for $0 < \varepsilon \le \bar{\varepsilon}$, $t \in [t_0, \bar{t}]$,

$$|\bar{y} - z| \le c_2 \left(\varepsilon \sup_{t_0 \le t \le \bar{t}} |\varphi(t, \varepsilon)| \cdot |t - t_0| + |y_0 - z_0|\right).$$

Define the interval $[t_0, t_1(\varepsilon)]$: $t_1 > t_0$, $t_1 - t_0 \le K/\varepsilon$, on which for $t \in [t_0, t_1]$ the solution of (4) with initial point x_0, y_0, t_0 does not leave $G_1(\varepsilon)$.

Theorem 1. *Under conditions 1)–15), for arbitrary $K>0$, $\delta>0$, there exists $\varepsilon_1>0$ ($\varepsilon_1 \le \varepsilon_0$) such that for $0<\varepsilon\le\varepsilon_1$, $t\in[t_0, t_1(\varepsilon)]$,*

$$1)\ |y-\bar{y}|\le\delta; \qquad 2)\ |x-\bar{x}-\varepsilon u_1(\bar{x},\bar{y},t)|\le\varepsilon\delta,$$

where x, y are solutions of (1) with initial point x_0, y_0, t_0.

From conditions 1)–15) it follows that in G there exists a continuously differentiable solution of the equation

$$\frac{\partial u_2}{\partial t}+\frac{\partial u_2}{\partial y}Y_0=P-A_2, \qquad u_2(x,y,t)\big|_{x,y,t\in M}=0,$$

which is easily constructed from the characteristics, as is u_1.

Suppose the following additional conditions are satisfied:

16) In G, v_1, u_2 are bounded and have continuous bounded derivatives with respect to x, y, t.

17) The expressions

$$\int_{t_0}^{t_0+T} S\,dt, \qquad \int_{t_0}^{t_0+T}(P-A_2)\,dt, \qquad \int_{t_0}^{t_0+T} D^{-1}\left(Y_1+\frac{\partial Y_0}{\partial x}u_1-B_1\right)dt$$

(the integrals are taken along (3)) are uniformly bounded for $0\le T<\infty$.

If conditions 16), 17) are fulfilled, improved estimates hold:

Theorem 2. *Under conditions 1)–17), for arbitrary $K>0$ there exist $C>0$, $\varepsilon_1>0$ ($\varepsilon_1\le\varepsilon_0$) such that for $0<\varepsilon\le\varepsilon_1$, $t\in[t_0,t_1(\varepsilon)]$,*

$$1)\ |y-\bar{y}|\le C\varepsilon; \qquad 2)\ |x-\bar{x}-\varepsilon u_1(\bar{x},\bar{y},t)|\le C\varepsilon^2.$$

Remark 1. Restriction 3) can be weakened by allowing, as was done in (^{3}), multiple intersections of (3) with M. Then Theorems 1 and 2 remain valid with some modification of the formulations and conditions.

Remark 2. One may dispense with condition 15). Then, considering the solutions (1) and (4) on an interval $[t_0, t_2(\varepsilon)]\subseteq[t_0, K/\varepsilon]$ such that, for $t\in[t_0,t_2]$, the solutions (1), (4) lie in $G_1(\varepsilon)$, one may assert that inequalities 2) of Theorems 1 and 2 (with some modification of the conditions) remain valid; however, the function y entering the statements of the theorems is then no longer a solution of (4), but is defined as follows:

$$\bar{y}\big|_{t=t_0}=y_0, \qquad |\bar{y}-Y_0(\bar{x},\bar{y},t)-\varepsilon B_1(\bar{x})|\le\varepsilon\omega,$$

where $\omega = \alpha(\varepsilon)$ for Theorem 1 and $O(\varepsilon)$ for Theorem 2, while $\bar{x}$ is the solution of (4).

An asymptotic method connected with an averaging scheme different from (^{3}) and having another domain of application was proposed in (^{4}).

§ 3. **Systems in standard form.** In (^{1}) systems in standard form $\dot{x} = \varepsilon X(x, t)$, which are a special case of (1), were studied. From the results of (^{3}) follow the approximation equations derived in (^{1}); from (4), in particular, follow the equations of the second approximation

$$\dot{\bar{x}} = \varepsilon X_0(\bar{x}) + \varepsilon^2 M_t \left\{ \left(\widetilde{X} \frac{\partial}{\partial x} \right) X(\bar{x}, t) \right\},$$

where $\partial \widetilde{X} / \partial t = X(\bar{x}, t) - X_0(\bar{x})$, $X_0 = M_t X$, and M_t is the averaging operator with respect to the explicitly occurring t.

§ 4. **Systems with a rapidly rotating phase.** In (^{4}) systems with a rapidly rotating phase, which are a special case of (1), were studied; they can be written in the form $\dot{x} = \varepsilon A(x, \psi, \varepsilon)$, $\dot{\psi} = \omega(x) + \varepsilon B(x, \psi, \varepsilon)$, where A, B are periodic in ψ. From the results of (^{3}) follow the approximation equations derived in (^{2}); from (4), in particular, follow the equations of the second approximation for x, ψ.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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