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Abstract

Full Text

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SOME PROPOSITIONS ON THE REGULARIZATION OF LINEAR OPERATORS IN BANACH SPACES

(Presented by Academician V. I. Smirnov on 19 V 1961)

1. Let E and E_1 be Banach spaces and let A be a closed linear operator from E into E_1 . Denote by $D(A)$ the domain of definition of the operator; by $\mathfrak{R}(A)$, the range of its values; by Z_A , the subspace of its zeros; and by Z_A^* , the subspace of all linear functionals orthogonal to $\mathfrak{R}(A)$. Following ⁽¹⁾, we shall call the operator A a Φ -operator (a Φ_+ -operator) if it is normally solvable* and the subspaces Z_A and Z_A^* are finite-dimensional (Z_A is finite-dimensional, while Z_A^* is not).

We shall give here several known propositions (see ⁽¹⁾) that will be needed below. If x_1, x_2, \dots, x_n are linearly independent elements of the Banach space E , and f_1, f_2, \dots, f_n are functionals biorthogonal to them, then the space E decomposes into the direct sum $E = \sigma_1 + \sigma_2$, where σ_1 is the finite-dimensional subspace with basis x_1, x_2, \dots, x_n , and σ_2 is the subspace of common zeros of the functionals f_1, f_2, \dots, f_n . To verify the validity of this assertion, it is enough to note that for any $x \in E$ the element

$$y = x - \sum_{i=1}^n f_i(x)x_i \in \sigma_2$$

and that, if $y_1 \in \sigma_1$, $y_2 \in \sigma_2$ and $y_1 + y_2 = 0$, then $y_1 = y_2 = 0$, since then $f_i(y_1) = 0$ for all i . Suppose now that A is a Φ_+ -operator. In this case the subspace $\overline{D(A)}$ is representable in the form of the direct sum $\overline{D(A)} = Z_A + \sigma$, where σ is the intersection of the subspace of common zeros of the functionals biorthogonal to a basis of the subspace Z_A with the subspace $\overline{D(A)}$. The operator A , considered as an operator from $\sigma \cap D(A)$ into $\mathfrak{R}(A)$, has an inverse operator R_0 . Consequently, it is bounded as an operator from $\mathfrak{R}(A)$ into E . If x_1, x_2, \dots, x_n is a basis in Z_A and f_1, f_2, \dots, f_n are functionals biorthogonal to this basis, then

$$R_0Ax = x - \sum_{i=1}^n f_i(x)x_i.$$

2. As usual, by a regularizer of the operator A we shall mean a bounded operator R from E_1 into E such that $RA = I + T$, where I is the identity and T is a completely continuous operator in E .

Theorem 1. *The difference of two regularizers of a Φ -operator is a completely continuous operator.*

Proof. Let f_1, f_2, \dots, f_n be a normalized basis of the subspace Z_A^* , and let y_1, y_2, \dots, y_n be elements of the space E_1 biorthogonal to the indicated basis. According to what was said in the preceding paragraph,

$$E_1 = \mathfrak{R}(A) + \sigma_1, \quad (1)$$

—

* This is equivalent to the closedness of the linear manifold $\mathfrak{R}(A)$.

where σ_1 is a subspace with basis y_1, y_2, \dots, y_n . The decomposition indicated above guarantees the existence of bounded projectors $P_{\mathfrak{R}(A)}$ and P_{σ_1} onto the subspaces $\mathfrak{R}(A)$ and σ_1 , respectively.

Let now R_1 and R_2 be two regularizers of the operator A , and let $R = R_1 - R_2$. The operator RA is defined on all of E and is completely continuous. Therefore the operator R' , induced by the operator R on the subspace $\mathfrak{R}(A)$, is completely continuous (since $R'y = RAR_0y$).

The operator $R''y$, induced by the operator Ry on σ_1 , is also completely continuous, since it is bounded and the subspace σ_1 is finite-dimensional. Since

$$R = R'P_{\mathfrak{R}(A)} + R''P_{\sigma_1},$$

the operator R is also completely continuous.

Corollary 1. The totality of all regularizers of a bounded Φ -operator A is described by the formula

$$R = R^0 + T, \quad (2)$$

where R^0 is an arbitrary fixed regularizer of the operator A , and T is an arbitrary completely continuous operator from E_1 into E .

Corollary 2. The totality of all regularizers of a closed Φ -operator is described by the same formula (2), but the completely continuous operator T must satisfy the additional condition: the operator TA is completely continuous.

Corollary 3. The totality of all regularizers of a Φ_+ -operator is described by the same formula (2), in which R^0 has its former meaning, and T is a bounded operator from E_1 into E , inducing on $\mathfrak{R}(A)$ a completely continuous operator T' such that the operator $T'A$ is completely continuous.

3. It was shown by S. G. Mikhlin ⁽²⁾ that if an operator A admits a regularization, then it is a Φ_+ -operator. With the aid of Theorem 1 one can obtain the following result in this direction:

Theorem 2. *A bounded operator A from E into E_1 admits a regularization (on the left) if and only if: 1) it is a Φ_+ -operator; 2) there exists a completely continuous operator T in the space $\mathfrak{R}(A)$ such that the operator $I_{\mathfrak{R}(A)} + T$ admits an extension to a bounded operator from E_1 into $\mathfrak{R}(A)$ (by I_σ , here and below, is denoted the identity operator of the space σ). If the operator A is only closed,*

then for the existence of a regularizer it is sufficient that conditions 1 and 2 and the condition be fulfilled: the operator TA is completely continuous.

Corollary. If A is a Φ_+ -operator and $\mathfrak{R}(A)$ has a direct complement in E_1 , then the operator A admits a regularization.

In particular, the conditions of the corollary are fulfilled if A is a Φ -operator or if the space E_1 is Hilbert (cf. ^(3,4)).

Up to now we have considered only bounded regularizers. If one does not assume boundedness of the regularizer, then the following holds:

Theorem 3. In order that a closed operator admit a regularization, it is necessary and sufficient that it be a Φ_+ -operator.

4. It was shown by S. G. Mikhlin ⁽²⁾ that if E is a Φ -operator acting in Hilbert space, then the equation $A\varphi = f$, in the case of its solvability, is equivalent to a certain Riesz-Schauder equation of the form $RA\varphi = Rf$. It turns out that this assertion also holds in arbitrary Banach spaces.

Lemma. Let A and R be distributive operators and let $D(R) \supseteq \mathfrak{R}(A)$. In order that the equations $A\varphi = f$ and $RA\varphi = Rf$ be equivalent for elements f of some set M , it is necessary and sufficient that the equation $R\psi = 0$ have no nontrivial solutions of the form $\psi = \psi_1 - \psi_2$, where $\psi_1 \in \mathfrak{R}(A)$, and $\psi_2 \in M$.

Proof. Necessity. Let the equations $A\varphi = f$ and $RA\varphi = Rf$ be equivalent for all $f \in M$. Suppose that the equation $R\psi = 0$ has a nontrivial solution ψ_0 of the indicated form, i.e.

$$\psi_0 = A\varphi_0 - f_0, \quad \text{where } f_0 \in M.$$

Two cases are possible:

- 1) $f_0 \notin \mathfrak{R}(A) \cap M$. In this case the equation $A\varphi = f_0$ is unsolvable, whereas the equation $RA\varphi = Rf_0$ has the solution φ_0 .
- 2) $f_0 \in \mathfrak{R}(A) \cap M$. In this case $\psi_0 = A\varphi_0 - A\varphi_1 \neq 0$, so that $A\varphi_0 \neq A\varphi_1$. The equation $RA\varphi = RA\varphi_1$ has the solution φ_0 , which is not a solution of the equation $A\varphi = A\varphi_1$.

In both cases we arrive at a contradiction, which proves the necessity of the condition of the lemma.

Sufficiency. Suppose that the conditions of the lemma are fulfilled and $f \in M$. Consider the equation $RA\varphi = Rf$. It is equivalent to the equation $R(A\varphi - f) = 0$. If this equation has a solution φ_0 , then $A\varphi_0 = f$, since the equation $R\psi = 0$ has no nontrivial solutions of the form $A\varphi_0 - f \in M$.

Corollary. In order that the equations $A\varphi = f$ and $RA\varphi = Rf$ be equivalent, in the case of solvability of the first of them it is necessary and sufficient that the equation $R\psi = 0$ have no nontrivial solutions in the range of the operator

A . In order that the indicated equations be equivalent for all f , it is necessary and sufficient that the equation $R\psi = 0$ have no nontrivial solutions.

Theorem 4. *If A is a Φ_+ -operator, then the equation $A\varphi = f$, in the case of solvability, is equivalent to a certain Riesz-Schauder equation.*

Proof. Let R_0 be the operator constructed in § 1. This operator is initially defined only on $\mathfrak{R}(A)$. In the usual way we extend the operator R_0 to a distributive operator R (generally speaking, unclosed) defined on the whole space E_1 . By construction, the extended operator R has no zeros in $\mathfrak{R}(A)$. Therefore the equations $A\varphi = f$ and $RA\varphi = Rf$ are equivalent for $f \in \mathfrak{R}(A)$. It remains to observe that

$$RA\varphi = R_0A\varphi = \varphi \sum_{i=1}^{\dim Z_A} f_i(\varphi)x_i,$$

i.e., that RA is a Riesz-Schauder operator.

Corollary 1. *If the space E_1 decomposes into the direct sum of its subspaces $\mathfrak{R}(A)$ and σ_1 , then the equation $A\varphi = f$, in the case of its solvability, can be reduced to an equivalent Riesz-Schauder equation by the application of a bounded regularizer.*

Indeed, putting

$$R\varphi = \begin{cases} R_0\varphi, & \varphi \in \mathfrak{R}(A), \\ 0, & \varphi \in \sigma_1, \end{cases}$$

we obtain a bounded extension of the operator R_0 to the whole space E_1 . The decomposition indicated in this corollary occurs, for example, if E_1 is a Hilbert space.

Corollary 2. *If A is a Φ -operator, then the equation $A\varphi = f$, in the case of its solvability, reduces to an equivalent Riesz-Schauder equation by the application of a bounded regularizer.*

From Corollary 2, in particular, follows the theorem of S. G. Mikhlin mentioned at the beginning of this section.

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