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MODULES OVER A K-SPACE

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1. The abstract norming of spaces of type B_K (^{1}) makes it possible to study mathematical objects by means of functional analysis in greater detail than in spaces of type B. Here modules over a K-space with unit (^{1}) are constructed; one of them is a generalization of a real Banach space and, in particular, has the properties of a space of type B_K; another is a generalization of a real Hilbert space.

2. A K-space as a generalized ring. A K-space Z with unit is transformed in various ways ($^{1-3}$), equivalent to one another (as follows from Theorem 1 (^{3})), into a commutative generalized ring (the product zu exists not for all $z, u \in Z$), in which for all elements z the quantity $\sqrt[n]{z}$ is defined and for some an inverse z^{-1} is defined ($zz^{-1} = e_z$, where e_z is the trace of the element z (^{1,2})). The maximal extension $\widetilde{Z}$ (^{1}) of the K-space Z is an absolute field (with the same unit): the product $\tilde{z}\tilde{u}$ and the inverse $\tilde{z}^{-1}$ exist for all $\tilde{z}, \tilde{u} \in \widetilde{Z}$ (^{2}). For simplicity of notation we shall assume that $Z \subset \widetilde{Z}$. Let us note one property of multiplication in Z. For disjoint $z_i \in Z$, $z_i \geq 0$, there exists $\sqrt{z_1}\sqrt{z_2}$, since (^{2,3})

$$z_i \leq z = z_1 \vee z_2$$

and $\sqrt{z}\sqrt{z}$ exists in Z.

3. A quotient in a K-space.
Definition 1. If the result of the action zu^{-1} in $\widetilde{Z}$ on elements from Z belongs to Z, we shall call it the **quotient** z/u.

Let us note some properties of the quotient, immediately following from the properties of the actions zu^{-1} in $\widetilde{Z}$ and the embedding of Z in $\widetilde{Z}$ ($^{1-3}$). The operation of division is normal, i.e., if $|z| \leq |v|$, $|u| \geq |w|$, and v/w exists, then z/u also exists, and moreover $|z/u| \leq |v/w|$. In order that $v = z/u$, it is necessary and sufficient that in Z

$$vu = ze_u, \qquad e_v = e_z \wedge e_u.$$

If u^{-1} and zu^{-1} exist in Z, then $zu^{-1} = z/u$. The operation of division is related in the usual way to congruence relations, arithmetic operations (provided multiplication or division of the given elements can be carried out), and extraction of a root of natural degree. If udv (^{1}), then

$$\frac{z}{u+v} = \frac{z}{u} + \frac{z}{v}.$$

Any circumstance will henceforth be called **normal** (cf. (^{1})) if, from the fact that it holds for some $u \in Z$, it follows that it is preserved for every $z \in Z$ with modulus $|z| \leq |u|$.

4. Convergence in a K-space. In a K-space Z one can establish several kinds of convergence. Suppose that in Z convergence $z_n \to 0$ is defined in some way for certain $z_n \geq 0$. Put $z_n \to z$ if $|z_n - z| \to 0$.

We shall require that convergence satisfy the conditions: 1) addition is continuous (with respect to a countable sequence); 2) convergence is normal: if $0 \leq z_n \leq u_n \to 0$, then $z_n \to 0$; 3) if $z_n = z$, $z_n \to 0$, then $z = 0$.

Hence the uniqueness of the limit follows. The indicated properties are possessed, for example, by (o)-convergence (^{1}); by (bk)-convergence, if Z is a space of type B_K with a monotone norm: from $|z| \leq |u|$ it follows that $\|z\| \leq \|u\|$.

in the K-space W of norms; topological convergence, if Z is a KT-space (^{4}) with the first separation axiom. For such convergence theorem 2 (^{4}) remains valid.

5. A generalized module over a K-space

Elements of Z and $\widetilde{Z}$ shall be called **scalars**.

Definition 2. Let, for some elements of Z and some elements of the additive commutative group X, the products $zx, xz \in X$ be uniquely defined, and let $1x = x$ and, for $z, u \in Z,\ x, y \in X$,

$$zx = xz, \quad z(x+y) = zx + zy, \quad (z+u)x = zx + ux, \quad z(ux) = (zu)x \qquad (1)$$

provided that the products exist at least in one part of each equality (1); multiplication by a scalar is normal: if ux exists in X, $|z| \leq |u|$, then zx also exists in X. Then we shall call X an M_z-module.

If λ is a real number, then by λx we mean $\lambda 1x$.

Definition 3. If Z is an absolute field (see above) and in an M_z-module X the product zx exists for all $z \in Z,\ x \in X$, then we shall call X an **absolute M_z-module**.

Definition 4. An absolute $M_{\widetilde{Z}}$-module $\widetilde{X} \supset X$ will be called an M**-extension** of the M_z-module X, if the meaning of the product zx for elements $z \in Z,\ x \in X$ is the same in X and in $\widetilde{X}$ (provided that zx exists in X). Here, instead of an isomorphism, the inclusion $X \subset \widetilde{X}$ is adopted also merely for simplicity of notation in what follows.

Theorem 1. *For every M_z-module there exists its M-extension.*

Construct $\widetilde{X}$ from all sequences $\tilde{x} = \{\tilde{x}_n\}$ $(n = 1, \dots, \infty)$ of all possible "linear" combinations

$$\tilde{x}_n = \sum_{k=1}^{m} z_{kn} x_k,$$

where m is any non-fixed natural number; $x_k \in X,\ x_i \neq x_k$ for $i \neq k$; for each k, $\{z_{kn}\}$ is a (o)-convergent sequence in $\widetilde{Z}$ of finite-valued elements $(^1)$. We shall consider elements $\tilde{x}$ and $\tilde{y} = \{\tilde{y}_n\} \in \widetilde{X}$ (where $\tilde{y}_n = \sum_{i=1}^{p} u_{in} y_i$) equal if $m = p,\ x_k = y_k,\ (o)\!-\lim_{n\to\infty} z_{kn} = (o)\!-\lim_{n\to\infty} u_{kn}$ $(k = 1, \dots, m)$.

To an element $x \in X$ we put in correspondence the stationary sequence $\{x\} \in \widetilde{X}$. Identifying x with $\{x\}$, we obtain the inclusion $X \subset \widetilde{X}$. Addition in $\widetilde{X}$ is defined "coordinatewise" : $\tilde{x} + \tilde{y} = \{\tilde{x}_n + \tilde{y}_n\}$, by "combining like terms" so that all x_k, y_i are distinct. Define multiplication by a scalar:

$$zx = xz = \left\{ \sum_{k=1}^{m} z_n z_{kn} x_k \right\},$$

where the finite-valued $z_n \xrightarrow{(o)} z$ (the existence of such z_n is evident from lemma 2.17 of Chapter 4 in $(^1)$). Using the (o)-continuity of addition and multiplication $(^1)$ in $\widetilde{Z}$, it is not difficult to verify that all conditions of definition 4 are fulfilled.

6. A module over a space of norms

Definition 5. Let to each element x of an M_z-module X there be assigned a norm—an element $\|x\| \in Z$, with: 1) if $\|x\| = 0$, then $x = 0$; 2) $\|x+y\| \leq \|x\| + \|y\|$; 3) $\|zx\| = |z|\|x\|$, if zx or $z\|x\|$ exists. Then we shall call X a B_z-module.

As usual, we establish that $\|0\| = 0,\ \|x\| \geq 0$. If $|z| \leq |u|$, ux exists in X, then $\|zx\| \leq \|ux\|$.

Definition 6. The **trace** e_x of an element x is the trace $e_{\|x\|}$ of its norm. If $e_x d e_y$, i.e. $\|x\| d \|y\|$, then x and y will be called **disjoint**: xdy. If $e_z d e_x$, i.e. $zd\|x\|$, then we shall call zdx.

If zdx, then $zx = 0$, and therefore $e_x x = x$. Indeed, $x = (e_x + Ce_x)x$, where $Ce_x = 1 - e_x$; $Ce_x dx$. If xdy, then $\|x \pm y\| = \|x\| + \|y\|$, since $\|x \pm y\| \geq \|x\| - \|y\| = \|x\| + \|y\|$ when $\|x\| d \|y\|$.

In X we define convergence: $x_n \to x$, if $\|x_n - x\| \to 0$ in Z in the sense of item 4 of the present work. It is not difficult to establish the usual properties of convergence in norm: uniqueness of the limit, continuity of the norm, etc.

If $x_i d x_k$, then for a finite sum or series

$$\left\| \sum_n x_n \right\| = \sum_n \|x_n\|.$$

This also holds for transfinite series, if convergence in Z is defined in the corresponding way.

Let us add the requirement of completeness of X with respect to convergence in norm.

Definition 7. A countably complete B_Z-module X will be called a **module of type** B_Z. If here convergence in Z is understood as (o)-convergence (or (bk)-convergence, see item 4), then X will be called a **module of type** $B_{\bar{K}}$ (or, respectively, B_{BK}). A module of type B_K is a special case of a module of type B_{BK} (for $Z = W$, i.e. $\|z\| = |z|$, see item 4).*

Theorem 2. *A module of type B_K is a space of type B_K.*

Axiom 4 (^{1}) of a space of type B_K is fulfilled in any B_Z-module: let

$$\|x\| = z_1 + z_2, \quad z_i \geqslant 0;$$

take

$$x_i = \frac{z_i}{z_1 + z_2} x,$$

then

$$\|x_i\| = z_i(e_{z_1} \vee e_{z_2}) = z_i, \qquad x_1 + x_2 = e_{xx} = x.$$

Definition 8. If a B_Z-module is an absolute M_Z-module, then we shall call it an **absolute B_Z-module**. Such a module of type B_Z will be called **absolute**.

Definition 9. An absolute $B_{\bar{Z}}$-module $\widetilde{X}$ will be called a B**-extension** of the B_Z-module X, if $\widetilde{X}$ is an M-extension of the $M_{\bar{Z}}$-module X and the meaning of the norm $\|x\|$ for $x \in X$ is the same in X and $\widetilde{X}$.

Theorem 3. *For every B_Z-module there exists its B-extension.*

Define in the M-extension $\widetilde{X}$ the norm

$$\|\tilde{x}\| = (o)\text{-}\lim_{n \to \infty} \|\tilde{x}_n\|.$$

Then $\widetilde{X}$ satisfies all the requirements of Definition 9.

7. Module over a space of scalar products

Definition 10. Let to each pair x, y of elements of the M_Z-module X there correspond a scalar product $(x, y) \in z$, with: 1) $(x, x) \geqslant 0$; if $(x, x) = 0$, then $x = 0$; 2) $(x, y) = (y, x)$; 3) $(x + y, y') = (x, y') + (y, y')$; 4) $(zx, y) = z(x, y)$, if zx or $z(x, y)$ exists. We shall call X an H_Z-module.

Definition 11. If an H_Z-module is an absolute M_Z-module, then we shall call it an **absolute H_Z-module**. Such a module of type H_Z (see Definition 13) will be called **absolute**.

Definition 12. An absolute $H_{\widetilde{Z}}$-module $\widetilde{X}$ will be called an **H-extension** of the H_Z-module X, if $\widetilde{X}$ is an M-extension of the M_Z-module X and the meaning of (x, y) for $x, y \in X$ is the same in X and $\widetilde{X}$.

Theorem 4. *For every H_Z-module there exists its H-extension.*

Define in the M-extension $\widetilde{X}$ the scalar product

$$(\tilde{x}, \tilde{y}) = (o)\text{-}\lim_{n\to\infty} (\tilde{x}_n, \tilde{y}_n).$$

All the conditions of Definition 12 are fulfilled.

Define in X, as usual, the norm

$$\|x\| = \sqrt{(x, x)}.$$

Then $e_x = e_{(x,x)}$. As before it is proved that $e_{xx} = x$. Therefore

$$(x, y)e_y = (x, e_{yy}) = (x, y).$$

Theorem 5. *For an H_Z-module X relation (2) is fulfilled; X is a B_Z-module.*

Let $x, y \in X$. In $\widetilde{Z}$ there exists

$$\tilde{z} = (x, y)/(y, y).$$

Construct in the H-extension $\widetilde{X}$ the element $x - \tilde{z}y$. For the $H_{\widetilde{Z}}$-module $\widetilde{X}$, in $\widetilde{Z}$ the relations

$$0 \leqslant (x - \tilde{z}y,\ x - \tilde{z}y) = (x, x) - (x, y)^2/(y, y)$$

are fulfilled. Multiply by (y, y). Then

$$(x, x)(y, y) - (x, y)^2 e_y \geqslant 0, \qquad (x, x)(y, y) \geqslant (x, y)^2 \geqslant 0.$$

Since extraction of the root is an increasing operation ([1],[2]), it follows from this the comparability in Z

$$|(x, y)| \leqslant \|x\|\|y\|. \tag{2}$$

Verification of the axioms of a B_Z-module is carried out in the usual way. It is clear that an absolute H_Z-module is an absolute B_Z-module and that an H-extension is simultaneously also a B-extension, if in $\widetilde{X}$ one sets $\|\tilde{x}\| = \sqrt{(\tilde{x}, \tilde{x})}$ (taking into account the (o)-continuity of the root $(^1)$).

From comparability (2) it follows that $e_{(x,y)} \leq e_x \wedge e_y$; if xdy, then $x \perp y$.

Definition 13. A countably complete H_Z-module X will be called a **module of type** H_Z. If here convergence in Z is understood as (o)-convergence (or (bk)-convergence, see item 4), then X will be called a **module of type** H_K (or, respectively, H_{B_K}).

Theorem 6. *A module of type H_Z (or respectively H_K, H_{B_K}) is a module of type B_Z (respectively B_K, B_{B_K}). A module of type H_K is a space of type B_K.*

A module of type H_K is also a special case of a module of type H_{B_K}.

Definition 14. A countable system of elements e_n of an H_Z-module will be called **orthonormal** if $e_i \perp e_k$, $\|e_n\| = e_n$, where e_n are unit elements from the base (|) of the K-space Z.

As usual, the system $\{e_n\}$ is called **complete** in X if from $x \perp e_n$ for all n it follows that $x = 0$.

Theorem 7. *If in a module X of type H_K there is a complete orthonormal system $\{e_n\}$ $(n = 1, \dots, \infty)$, then for every element $x \in X$ the expansion*

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n, \qquad \|x\|^2 = \sum_{n=1}^{\infty} (x, e_n)^2$$

holds.

Let us note that here axiom 5 $(^1)$ of the K-space Z is used only for countable sets.

If the set of all norms $\|x\|$ coincides with Z, then from the completeness of $\{e_n\}$ in X follows the completeness of $\{e_n\}$ in Z, i.e. $\sup\{e_n\} = 1$. In particular, disjoint elements may be taken as the $\{e_n\}$, but the completeness condition of $\{e_n\}$ is still expressed through orthogonality.

8. Examples. Let X and Z consist of real bounded almost everywhere functions $x(s,t)$ and $z(s)$, square-summable (x on E, z on the projection S onto the s-axis of the set E). We naturally make X into a module over Z in the sense of Definition 2. Let m disjoint measurable sets S_i have union S; the sets $T(s)$ are all measurable sections for $s = \text{const}$ of the set E. Put

$$(x, y) = \int_{T(s)} x(s,t) y(s,t) \, dt, \qquad \|z\| = \left\{ \operatorname*{vrai\,sup}_{S_i} |z(s)| \right\} \quad (i = 1, \dots, m).$$

Then X becomes a module of type H_{B_K}, but not an absolute one. The proof of the countable completeness of X is similar to the proof of the countable completeness of the space L^2.

If, however, X consists of functions $x(t)$ square-summable on the set E_1, and Z consists of vectors $z = \{\zeta_1, \dots, \zeta_m\}$ (m fixed), e_i are measurable, $e_i \cap e_k = \Lambda$,

$$\bigcup_{i=1}^{m} e_i = E_1,$$

and the products zx and (x, y) are defined in the following way:

$$(zx)(t) = \zeta_k x(t) \quad \text{for } t \in e_k, \qquad (x, y) = \left\{ \int_{e_k} xy \, dt \right\}_{k=1,\dots,m},$$

then X becomes an absolute module of type H_K (convergence in Z is coordinatewise).

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Note: Figure translations are in progress. See original paper for figures.

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