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Abstract

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MATHEMATICS

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ON ONE REPRESENTATION OF A TENSOR FUNCTION

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A representation of a tensor function is proposed which is a generalization of the known representation of a vector function in three-dimensional space $\mathbf{a} = \operatorname{grad}\varphi_1 + \varphi_2 \operatorname{grad}\varphi_3$.

Let a vector field $\mathbf{a} = \mathbf{a}(x^1, x^2, x^3)$ be given in the three-dimensional space x^1, x^2, x^3. The vector $\mathbf{a}$ can be represented in the form (^{1})

$$a_i = \frac{\partial\varphi_1}{\partial x^i} + \varphi_2\frac{\partial\varphi_3}{\partial x^i}, \qquad i = 1, 2, 3, \tag{1}$$

where a_i are the covariant components of the vector $\mathbf{a}$; $\varphi_1, \varphi_2, \varphi_3$ are scalar functions of the coordinates x^1, x^2, x^3.

Let us consider the derivation of this formula, based on the use of the theory of Pfaffian forms. To an arbitrary three-dimensional vector $\mathbf{a}$ there corresponds the Pfaffian form

$$\omega = a_i dx^i, \tag{2}$$

where the summation index i takes the values $1, 2, 3$. It is known (^{2}) that the form (2), depending on whether it is of the first, second, or third class, is reduced to one of the following canonical forms:

$$d\varphi_1, \qquad \varphi_1 d\varphi_2, \qquad d\varphi_1 + \varphi_2 d\varphi_3, \tag{3}$$

where $\varphi_1, \varphi_2, \varphi_3$ are independent quantities that are functions of the coordinates x^1, x^2, x^3.

The criterion that the form is of the first or second class is, respectively, the fulfillment of the condition $\operatorname{rot}\mathbf{a} = 0$ or $\mathbf{a}\cdot\operatorname{rot}\mathbf{a} = 0$. Consequently, these cases correspond to certain special vectors. For an arbitrary vector the conditions given above are not satisfied, and the corresponding form will be of the third class.

Thus, in the general case one may take

$$\omega = d\varphi_1 + \varphi_2 d\varphi_3. \tag{4}$$

If the differentials of the functions φ_1 and φ_3 are written out explicitly, it is easy to see that formula (1) follows from equalities (2) and (4).

Thus, the specification of three scalar functions determines a vector function. Formula (1) can be generalized to the case of n-dimensional vectors. Indeed, to an n-dimensional vector **a** there corresponds the Pfaffian form

$$\omega = a_\alpha dx^\alpha; \tag{5}$$

here the summation index α takes the values $1, 2, \dots, n$.

Denote by p the class of the form (5). From the theory of Pfaffian forms it follows (3) that in the case of an odd class $p = 2k + 1$ the form (5) can be reduced to the following canonical form:

$$\omega = d\varphi_1 + \varphi_2 d\varphi_3 + \dots + \varphi_{2k} d\varphi_{2k+1}, \tag{6}$$

and in the case of an even class $p = 2k$, in the form

$$\omega = \varphi_1\, d\varphi_2 + \varphi_3\, d\varphi_4 + \dots + \varphi_{2k-1}\, d\varphi_{2k}, \tag{7}$$

where φ_i are independent functions of the coordinates $x^1, x^2, \dots, x^n$, and always $p \le n$. The class of the form (5) is defined as the rank of the characteristic system of equations

$$a_\alpha dx^\alpha = 0, \qquad a_{\beta\alpha} dx^\alpha = 0, \qquad a_{\beta\alpha} = \frac{\partial a_\alpha}{\partial x^\beta} - \frac{\partial a_\beta}{\partial x^\alpha}, \quad \alpha, \beta = 1, 2, \dots, n. \tag{8}$$

In particular, if $a_{\beta\alpha} = 0,\ \alpha, \beta = 1, 2, \dots, n$, then the rank of the system (8) is equal to unity and the form (5) has the canonical form

$$\omega = d\varphi. \tag{9}$$

Let us turn to the equalities (6) and (7). Writing the differentials of the functions φ_i occurring in them in expanded form and comparing the results with equality (5), we find that

$$a_\alpha = \frac{\partial \varphi_1}{\partial x^\alpha} + \varphi_2 \frac{\partial \varphi_3}{\partial x^\alpha} + \dots + \varphi_{2k} \frac{\partial \varphi_{2k+1}}{\partial x^\alpha}, \quad \alpha = 1, 2, \dots, n, \tag{10}$$

in the case when the form (5) is of odd class $p = 2k + 1$, and

$$a_\alpha = \varphi_1 \frac{\partial \varphi_2}{\partial x^\alpha} + \varphi_3 \frac{\partial \varphi_4}{\partial x^\alpha} + \dots + \varphi_{2k-1} \frac{\partial \varphi_{2k}}{\partial x^\alpha}, \quad \alpha = 1, 2, \dots, n, \tag{11}$$

for a form of even class $p = 2k$.

The highest class of the form (5) coincides with the number n of dimensions of the space. Consequently, if n is odd, then the most general form of the vector **a** is given by formula (10), in which $2k + 1 = n$; for even n, by formula (11), where $2k = n$.

For a vector of special form whose components satisfy the conditions $a_{\beta\alpha} = 0$, $\alpha, \beta = 1, 2, \dots, n$, we have the representation

$$a_\alpha = \frac{\partial \varphi}{\partial x^\alpha}, \quad \alpha = 1, 2, \dots, n. \tag{12}$$

Consider a tensor H, which is a tensor function of a certain variable tensor T and of constant tensors $A_1, \dots, A_s$, playing the role of parameters (4)

$$H = F(G, T, A_1, \dots, A_s); \tag{13}$$

here G is the metric tensor. For definiteness we shall assume that H and T are symmetric tensors of the second rank. Consider the space in which the coordinates of a point are the contravariant components of the tensor T: $T^{11}, T^{22}, T^{33}, T^{12}, T^{23}, T^{31}$. In this space the tensor H may be regarded as a vector with covariant components $H_{11}, H_{22}, H_{33}, H_{12}, H_{23}, H_{31}$. In the general case this vector may, according to (11), be represented in the form

$$H_{kl} = \varphi_1 \frac{\partial \varphi_2}{\partial T^{kl}} + \varphi_3 \frac{\partial \varphi_4}{\partial T^{kl}} + \varphi_5 \frac{\partial \varphi_6}{\partial T^{kl}}, \quad kl = 11, 22, 33, 12, 23, 31, \tag{14}$$

where $\varphi_1, \dots, \varphi_6$ are scalar functions depending on the system of joint invariants of the tensors $G, T, A_1, \dots, A_s$.

Thus, specifying the tensor function (13) is equivalent to specifying 6 scalar functions.

If for the function (13) the conditions

$$\frac{\partial H_{kl}}{\partial T^{mn}} - \frac{\partial H_{mn}}{\partial T^{kl}} = 0, \quad k, l, m, n = 1, 2, 3, \tag{15}$$

among which 15 are independent, we shall have

$$H_{kl} = \frac{\partial \varphi}{\partial T^{kl}}, \qquad k, l = 1, 2, 3, \tag{16}$$

where φ is a scalar function of the joint invariants of the tensors $G, T, A_1, \dots, A_s$. The function (13) in this case is called potential. Comparing formulas (14) and (16), we see that an arbitrary function (13) can be regarded as a combination of 3 potential functions (16).

Let us turn to the case when the tensor function (13) is isotropic (i.e., all parametric tensors $A_1, \dots, A_s$ are spherical tensors). The scalar functions φ_i in (14) will depend only on the three independent invariants I_1, I_2, I_3 of the tensor T.

We have

$$I_1 = g_{\alpha\beta} T^{\beta\alpha}, \qquad I_2 = g_{\alpha k} g_{\beta l} T^{k\beta} T^{l\alpha}, \qquad I_3 = g_{\alpha k} g_{\beta l} g_{\gamma m} T^{k\beta} T^{l\gamma} T^{m\alpha}. \tag{17}$$

It is easy to verify the validity of the formulas

$$\frac{\partial I_1}{\partial T^{lk}} = g_{kl}, \qquad \frac{\partial I_2}{\partial T^{lk}} = 2T_{kl}, \qquad \frac{\partial I_3}{\partial T^{lk}} = 3T_{km} T^m_{\ l}. \tag{18}$$

For an isotropic tensor function, formula (14), taking into account the equalities (18), may be represented in the form

$$H_{kl} = k_0 g_{kl} + k_1 T_{kl} + k_2 T_{km} T^m_{\ l}, \tag{19}$$

where k_0, k_1, k_2 are functions of the invariants (17), determined by the equalities

$$\begin{aligned} k_0 &= \varphi_1 \frac{\partial \varphi_2}{\partial I_1} + \varphi_3 \frac{\partial \varphi_4}{\partial I_1} + \varphi_5 \frac{\partial \varphi_6}{\partial I_1}, \\ k_1 &= 2\left(\varphi_1 \frac{\partial \varphi_2}{\partial I_2} + \varphi_3 \frac{\partial \varphi_4}{\partial I_2} + \varphi_5 \frac{\partial \varphi_6}{\partial I_2}\right), \\ k_2 &= 3\left(\varphi_1 \frac{\partial \varphi_2}{\partial I_3} + \varphi_3 \frac{\partial \varphi_4}{\partial I_3} + \varphi_5 \frac{\partial \varphi_6}{\partial I_3}\right). \end{aligned} \tag{20}$$

Thus, specifying the isotropic tensor function (13) is equivalent to specifying three scalar functions k_0, k_1, k_2 of the invariants I_1, I_2, I_3. Formula (19) can also be obtained from other considerations (see (^{4})).

Above we considered the case when the tensors H and T are of second rank and symmetric. The cases when the tensors H and T are of arbitrary, but identical, rank are considered analogously. Thus, for any (nonsymmetric) tensors H and T of second rank we have

$$H_{kl} = \frac{\partial \varphi_1}{\partial T^{lk}} + \varphi_2 \frac{\partial \varphi_3}{\partial T^{lk}} + \varphi_4 \frac{\partial \varphi_5}{\partial T^{lk}} + \varphi_6 \frac{\partial \varphi_7}{\partial T^{lk}} + \varphi_8 \frac{\partial \varphi_9}{\partial T^{lk}} \tag{21}$$

i.e., the specification of the tensor function is determined in this case by the specification of nine scalar functions.

For H and T of third rank we obtain a formula analogous to formula (21), in which 27 scalar functions occur, and so on.

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Note: Figure translations are in progress. See original paper for figures.

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