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Abstract

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ON THE REGULARITY OF CONVEX SURFACES WITH A REGULAR METRIC IN LOBACHEVSKY SPACE

The present note is devoted to the following question: what degree of regularity of a convex surface in Lobachevsky space is guaranteed by the prescribed regularity of its intrinsic metric, if nothing is assumed in advance about the external form of the surface except convexity? For the case of convex surfaces in Euclidean and elliptic spaces this question was solved in the author' s works ([1], [2]). In the note ([3]) a solution of the question was outlined for any space of constant curvature. However, this solution cannot be regarded as satisfactory for the case of Lobachevsky space, since the class of surfaces considered is restricted by the condition of positivity of the Gaussian curvature, which is not natural for Lobachevsky space.

The requirement of positivity of the Gaussian curvature of the surface arose in the attempt simply to carry over the proof of the theorem in the Euclidean case ([1]) to the case of surfaces in Lobachevsky space, in connection with establishing a priori estimates for the normal curvatures of convex caps. Now these estimates are obtained without the assumption of positivity of the Gaussian curvature. As a result, the theorem on the regularity of a convex surface with a regular metric in Lobachevsky space assumes the following natural form.

Theorem. *If a convex surface in Lobachevsky space has a regular metric and the Gaussian curvature of the surface is greater than the curvature of the space, then the surface is regular. Namely, if the metric of the surface is differentiable k times ($k > 5$), then the surface is differentiable at least $k - 1$ times. If the metric is analytic, then the surface is analytic.*

One of the main points of the proof of this theorem is the establishment of estimates for the normal curvatures of a convex regular cap. In the course of the proof such estimates have to be obtained under two assumptions:
1) under the condition that the normal curvatures at the edge of the cap are bounded, and then one deals with uniform estimates of the normal curvature on the whole cap;
2) without any assumptions on the edge of the cap, and in this case it is required to obtain estimates of the normal curvature on the set of interior points uniformly separated from the plane of the edge of the cap.

In order to illustrate the method by which the question of estimates of the normal curvatures is solved without the assumption of positivity of the Gaussian curvature, let us consider the first, simpler problem. Thus, let ω be a small regular convex cap with edge plane σ in Lobachevsky space of curvature $k_R = -1$. Let the normal curvatures of the cap along the edge be bounded by the constant c_0, and let the angles of the tangent planes with the plane of the base of the cap be bounded by the constant $\vartheta_0 < \pi/2$. Denote by X an arbitrary point of the cap, by $\bar{\kappa}(X)$ the maximal normal curvature of the cap at the point X, and by $\vartheta(X)$ the angle formed by the perpendicular dropped···

from the point X to the plane σ, with the inner normal of the cap. Consider the function

$$\overline{w}(X) = \frac{\overline{\varkappa}(X)}{(\cos\vartheta(X))^{\mu}},$$

where μ is a certain positive constant, which will be determined later.

The function $\overline{w}(X)$ attains a maximum at some point X_0 of the cap. Here two possibilities may occur: 1) the point X_0 lies on the edge of the cap; 2) the point X_0 is an interior point of the cap. In the first case, obviously, $\overline{w} \leqslant c_0/(\cos\vartheta_0)^{\mu}$, and we obtain the estimate for the normal curvatures $\varkappa \leqslant c_0/(\cos\vartheta_0)^{\mu}$.

Consider the case where the point X_0 is interior. Introduce, in a neighborhood of the point X_0 on the surface of the cap, a semigeodesic coordinate net u, v, taking as the u-line passing through the point X_0 the geodesic in the direction of the maximal normal curvature, and as the v-line the perpendicular geodesic. As parameters u and v we take the arcs of these geodesics. Then the line element of the cap takes the form $ds^2 = du^2 + c^2dv^2$. At the point X_0, $c = 1,\ c_u = c_v = 0,\ c_{uu} = -k,\ c_{uv} = 0,\ c_{vv} = 0$, where k is the Gaussian curvature of the cap at the point X_0.

Let us now denote by $\varkappa(X)$ the normal curvature of the cap at the point X in the direction u, and introduce the function

$$w(X) = \frac{\varkappa(X)}{(\cos\vartheta(X))^{\mu}}.$$

Since $\varkappa(X) \leqslant \overline{\varkappa}(X)$, and $\varkappa(X_0) = \overline{\varkappa}(X_0)$, the function $w(X)$ also attains a maximum at the point X_0, and our problem reduces to estimating the value of this maximum.

If one takes in space an arbitrary plane σ' and denotes by z the distance of the point (u, v) of the cap ω from this plane, then z, as a function of u, v, satisfies the following equation, analogous to Darboux' s equation in the Euclidean case:

$$(r + \alpha)(t + \gamma) - (s + \beta)^2 + \delta = 0, \tag{*}$$

$$\alpha = -(1-p^2)\,\mathrm{th}\,z, \qquad \beta = pq\;\mathrm{th}\,z - \frac{c_u}{c}q,$$

$$\gamma = -(c^2-q^2)\,\mathrm{th}\,z + cc_{up} - \frac{c_v}{c}q,$$

$$\delta = -(1+k)(c^2-c^2p^2-q^2),$$

where p, q, r, s, t are the first and second derivatives of z with respect to u and v. The coefficients L, M, N of the second quadratic form of the surface admit the representations

$$L = \frac{r+\alpha}{\Delta}, \qquad M = \frac{s+\beta}{\Delta}, \qquad N = \frac{t+\gamma}{\Delta},$$

$$\Delta^2 = 1 - p^2 - \frac{q^2}{c^2},$$

and the normal curvature is

$$\varkappa(X) = \frac{r-(1-p^2)\,\mathrm{th}\,z}{(1-p^2-q^2/c^2)^{1/2}}.$$

Take as the plane σ' the tangent plane to the cap at the point X_0, and regard z as positive on the side on which the cap is situated. Then at the point X_0, $p = q = 0$, and since $M = 0$, we have $s = 0$ and $t = o(1/r)$.

For brevity, set

$$w = \chi\lambda, \qquad \lambda = 1/(\cos\vartheta)^{\mu}.$$

Then, taking into account that $w_u = w_v = 0$, at the point X_0 we shall have

$$r_u = w\left(\frac{1}{\lambda}\right)_u, \qquad r_v = w\left(\frac{1}{\lambda}\right)_v,$$

$$r_{uu} = \frac{w_{uu}}{\lambda} + w\left(\frac{1}{\lambda}\right)_{uu} - \frac{w}{\lambda}r^2 + r,$$

$$r_{vv} = \frac{w_{vv}}{\lambda} + w\left(\frac{1}{\lambda}\right)_{vv} - \frac{w}{\lambda}t^2 + t.$$

Substituting these derivatives into the equation obtained by differentiating $(*)$ twice with respect to u, we obtain

$$\frac{1}{\lambda}(w_{uu}t + w_{vv}r) - (1+k)\frac{\lambda_{uu}}{\lambda} - r^2\frac{\lambda_{vv}}{\lambda} + 2ku\frac{\lambda_u}{\lambda} + O(r^2) = 0,$$

where $O(r^2)$ is a quadratic trinomial in r with bounded coefficients. Since w attains a maximum at the point X_0 and, consequently, $w_{uu} \leq 0$, $w_{vv} \leq 0$, we have

$$-(1+k)\frac{\lambda_{uu}}{\lambda} - r^2\frac{\lambda_{vv}}{\lambda} + 2ku\frac{\lambda_u}{\lambda} + O(r^2) \geq 0. \qquad (**)$$

If, as the plane σ', we take the plane of the base of the cap σ, then λ can be represented in the form

$$\lambda = \frac{1}{(1 - h_u^2 - h_v^2/c^2)^{\mu/2}},$$

where h is the distance from the point (u, v) of the cap to the plane σ. It follows from this that the derivatives of λ which enter the inequality $(**)$ are expressed in terms of the derivatives of h. We find these derivatives by using the invariance of the coefficients of the second quadratic form of the surface ω with respect to spatial parametrizations, i.e. by means of the identities

$$\frac{r - (1-p^2)\,\mathrm{th}\,z}{(1 - p^2 - q^2/c^2)^{1/2}} = -\frac{h_{uu} - (1-h_u^2)\,\mathrm{th}\,h}{(1 - h_u^2 - h_v^2/c^2)^{1/2}}.$$

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Introducing the derivatives of λ into the inequality $(**)$ allows us to give it the following form at the point X_0:

$$-2\mu(1+k)r^2 + Ar^2 + Br + C \geq 0,$$

where A, B, C are bounded expressions, and A admits an estimate independent of μ. Multiplying this inequality by λ^2, we obtain

$$-2\mu(1+k)w^2 + AW^2 + O(w) \geq 0.$$

If now we choose μ so large that $-2\mu(1+k) + A < -1$, then the inequality takes the form

$$-w^2 + O(w) \geq 0.$$

It follows from this that w at the point X_0 cannot be too large, and this proves the existence of the estimate w_0 for it. The estimate of w is at the same time an estimate for $\varkappa$, since $\varkappa(X) \leqslant w(X_0)$.

The establishment of estimates for the normal curvatures strictly inside the cap, away from the plane of the edge, is obtained by an analogous consideration of the auxiliary function

$$\overline{w}(X) = \frac{\overline{\varkappa}(X)h(X)}{(\cos\vartheta(X))^{\mu}},$$

where $\overline{\varkappa}$ and ϑ have the same meanings as before, and $h(X)$ is the distance of the point X from the plane of the edge of the cap σ.

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REFERENCES

[1] A. V. Pogorelov, *Bending of Convex Surfaces*, 1951.
[2] A. V. Pogorelov, *Some Questions in the Theory of Surfaces in Elliptic Space*, Kharkov, 1960.
[3] A. V. Pogorelov, DAN, **122**, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.