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MATHEMATICS

V. F. VLASOV and A. F. TIMAN

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Abstract

Full Text

MATHEMATICS

V. F. VLASOV and A. F. TIMAN

ON A CERTAIN RELATION FOR INTEGRALS OF MODULI OF TRIGONOMETRIC POLYNOMIALS

(Presented by Academician S. N. Bernstein on 17 II 1961)

Let

$$T_n(x) = \frac{a_0^{(n)}}{2} + \sum_{\nu=1}^{n} a_\nu^{(n)} \cos \nu x \tag{1}$$

be a sequence of even trigonometric polynomials with real coefficients, and let $\rho_n(x)$ be a sequence of functions of bounded variation, defined on $[-\pi, \pi]$, for which

$$\operatorname*{Var}_{x\in[-\pi,\pi]} \rho_n(x) = 1.$$

Denote by

$$c_n = \int_{-\pi}^{\pi} e^{int}\, d\rho_n(t) \tag{2}$$

the corresponding diagonal sequence of Fourier–Stieltjes coefficients for $\rho_n(x)$.

In the present note we consider the integrals

$$L(T_n; \rho_n) = \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} T_n(x+t)\, d\rho_n(t) \right| dx \tag{3}$$

and establish one relation between the sequences of constants (3) and (2).

Theorem. *If the coefficients of the polynomials (1) form a convex or concave system of numbers*

$$\left(\Delta^2 a_k^{(n)} = a_k^{(n)} - 2a_{k+1}^{(n)} + a_{k+2}^{(n)} \le 0 \quad \text{or} \quad \Delta^2 a_k^{(n)} \ge 0;\ k = 0, 1, \dots, n-1;\ a_{n+1}^{(n)} = 0\right)$$

and, for some positive sequence $\varepsilon_n = O\left(\frac{1}{n}\right)$,

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$$\operatorname*{Var}_{t\in[-\varepsilon_n,\varepsilon_n]} \rho_n(t) \geq 1 - O\left(\frac{1}{n}\right), \tag{4}$$

then the equality

$$L(T_n;\rho_n) = |c_n| \int_{-\pi}^{\pi} |T_n(x)|\,dx + O\left(\max_{0\leq\nu\leq n} |a_\nu^{(n)}|\right) \tag{5}$$

holds, in which $O(1)$ *is a quantity uniformly bounded with respect to* n *over all polynomials* $T_n(x)$.

We note that condition (4), while not, generally speaking, necessary, is in a certain sense essential for the validity of relation (5). Without it, i.e. in the general case, the theorem stated would already be false. To see this, it suffices to put $a_\nu^{(n)} = 1$ $(\nu = 0, 1, \dots, n)$

and consider the sequence of functions

$$\rho_n(t) = \begin{cases} 0, & -\pi \leq t < -\dfrac{2k+1}{2n}\pi, \\ \dfrac{1}{2}, & -\dfrac{2k+1}{2n}\pi \leq t < \dfrac{2k+1}{2n}\pi, \\ 1, & \dfrac{2k+1}{2n}\pi \leq t \leq \pi, \end{cases}$$

where $k = k(n)$ is a nonnegative integer-valued function of n, increasing without bound together with n. In this case it is known (see (3), Lemma 1) that, as $n \to \infty$,

$$L(T_n;\rho_n) = \frac{1}{2}\int_{-\pi}^{\pi} \left| D_n\left(x + \frac{2k+1}{2n}\pi\right) + D_n\left(x - \frac{2k+1}{2n}\pi\right)\right| dx =$$

$$= \frac{4}{\pi}\ln\{K(n)+1\} + O(1) \to \infty, \qquad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

whereas the right-hand side of (5) is bounded.

Thus, for the validity of relation (5) the condition $\varepsilon_n = O\left(\frac{1}{n}\right)$ is necessary in the sense that, if it is not satisfied, then there exists a sequence of functions $\rho_n(t)$ satisfying (4) and such that relation (5) does not hold.

On the other hand, the example

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$$\rho_n(t) = \begin{cases} 0, & -\pi \le t < -\dfrac{\pi}{2n}, \\ \dfrac{1}{2}, & -\dfrac{\pi}{2n} \le t < \dfrac{\pi}{2n}, \\ 1, & \dfrac{\pi}{2n} \le t \le \pi, \end{cases}$$

$$T_n(x) = \frac{1}{2} + \sum_{\nu=1}^{m} \cos \nu x + \cos nx,$$

where $m = m(n)$ is an arbitrary integer-valued function of n for which

$$\lim_{n\to\infty} m(n) = \infty, \qquad \sup_{n\ge 1} \frac{m}{n} < 1,$$

shows that, for the validity of relation (5), condition (4) alone, without the assumption of convexity or concavity of the coefficient system of the polynomial $T_n(x)$, is still insufficient. Since in this case $c_n = 0$, the right-hand side of (5) is bounded, whereas (see (4), Theorem 11)

$$L(T_n;\rho_n) = \frac{1}{2}\int_{-\pi}^{\pi} \left| D_m\left(x + \frac{\pi}{2n}\right) + D_m\left(x - \frac{\pi}{2n}\right)\right| dx + O(1) =$$

$$= \frac{4}{\pi} \cos \frac{m\pi}{2n} \ln(m+1) + O(1),$$

i.e. $L(T_n;\rho_n) \to \infty$ as $n \to \infty$.

At the same time, there exist examples showing that relation (5) may sometimes hold also in cases not covered by the theorems, i.e., in fact it is valid for a broader class of sequences $T_n(x)$ and $\rho_n(x)$.

Let us also note that in some concrete cases the asymptotic estimate of the sequence $L(T_n;\rho_n)$ is known.

In particular, for

$$\rho_n(t) = \begin{cases} 0, & -\pi \le t < 0, \\ 1, & 0 \le t \le \pi \end{cases}$$

the equality (6) is valid (see also (7), p. 8. 2. 33)

$$L(T_n;\rho_n) = \int_{-\pi}^{\pi} |T_n(x)|\, dx = \frac{4}{\pi}\left|\sum_{\nu=0}^{n} \frac{a_\nu^{(n)}}{n-\nu+1}\right| + O\left(\max_{0\le\nu\le n} |a_\nu^{(n)}|\right). \qquad (6)$$

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From the theorem presented and equality (6) there follows directly a number of other results relating to particular sequences of polynomials $T_n(x)$ and functions $\rho_n(x)$ (see, for example, (5)).

Corollary. If $a_\nu^{(n)} = O(1)$, then, under the conditions of the theorem, the sequence of constants $L(T_n; \rho_n)$ is bounded if and only if

$$c_n = O\left(\left[1 + \left|\sum_{\nu=0}^{n} \frac{a_\nu^{(n)}}{n-\nu+1}\right|\right]^{-1}\right).$$

In special cases, for $a_\nu^{(n)} = 1$ ($\nu = 0, 1, \dots, n$), this corollary contains the known convergence conditions for summation processes of the S. N. Bernstein–Rogosinski type for Fourier series (see ($1^2, {}^4, {}^5$)).

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Note: Figure translations are in progress. See original paper for figures.

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