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**Abstract**

**Full Text**

**MATHEMATICS**

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## **THE CORRECTNESS OF HILBERT' S THEOREM ON SURFACES OF CONSTANT NEGATIVE CURVATURE**

*(Presented by Academician P. S. Aleksandrov, 10 X 1960)*

**1°.** In the theory of surfaces Hilbert' s theorem is well known, asserting that in three-dimensional Euclidean space there is no surface of constant negative curvature that is everywhere regular and complete in the sense of its intrinsic geometry. The present note is connected with a question that has repeatedly been raised by many geometers: to what extent is the condition of constancy of curvature essential here?

With the aim of making some progress on this question, we establish the **correctness** of Hilbert' s theorem. In the most general terms this means that:

- 1) there exist estimate relations between certain quantities pertaining to the metric of a surface of variable negative curvature and certain quantities pertaining to its extrinsic geometry in three-dimensional Euclidean space;
- 2) Hilbert' s theorem is obtained from these relations by a limiting passage.

A detailed formulation of this assertion will be given below (see § 5°).

**2°.** Wishing to exclude difficulties of a topological nature, not connected with the substance of the question, we shall assume the surfaces under consideration to be simply connected. We shall also agree not to specify further that we are dealing with regular surfaces in three-dimensional Euclidean space.

**3°.** Let us first consider the case of constant curvature. As is known, in the case  $K = -1$  the angle  $\omega$  between the asymptotic lines satisfies the equation  $\omega''_{uv} = \sin \omega$ , where  $u, v$  are natural asymptotic parameters. Consider an arbitrary asymptotic line, for example  $u = 0$ . Since  $\sin \omega > 0$ , on it there will be a point, for example  $(0; 0)$ , where  $\omega'_u > 0$  (or  $\omega'_u < 0$ , but then we change the orientation of the asymptotic lines). It follows from the equation that for small  $\varepsilon > 0$  in the region  $(-\varepsilon < u < +\varepsilon; v > 0)$  one will have  $\omega'_u \geq \text{const} > 0$ . Consequently, in the region  $(-\delta < u < +\delta, v > 0)$ ,  $0 < \delta < \varepsilon$ , one has  $\sin \omega \geq \text{const} = m > 0$ . Hence  $\omega'_u \geq mv$ , which for large  $v$  contradicts the inequalities  $0 < \omega < \pi$ . This proves Hilbert' s theorem, but in fact something stronger has been proved, since the whole surface is not used in these arguments. Let us agree to call an **asymptotic  $\varepsilon$ -strip** a part of some complete surface of negative curvature which consists of

all points of the surface lying at distance  $< \varepsilon$  from some asymptotic line. Then the result of the preceding arguments is expressed as follows: for no  $\varepsilon > 0$  is an asymptotic  $\varepsilon$ -strip of constant negative curvature possible. We shall call this assertion the strengthened Hilbert theorem.

4°. If in the formulation of the strengthened Hilbert theorem, instead of the condition of constancy of curvature, one requires that the curvature remain

within negative bounds, then an incorrect assertion is obtained. In fact, on the ordinary helicoid there exist asymptotic  $\varepsilon$ -strips (screw strips) whose Gaussian curvature is enclosed within negative bounds. On such strips  $\omega = \pi/2$ . Thus, the presence of negative bounds for the curvature by itself does not yet cause an excessively rapid change of the angle  $\omega$ , as in Hilbert's case. But, as will now be seen, a similar effect occurs if the Gaussian curvature changes over the surface in a certain sense slowly; moreover, it manifests itself the more strongly, the more slowly the curvature changes.

Item 5°. Consider a complete surface of curvature  $K \leq -1$ . Introduce the notation:  $k^2 = |K|$ ,  $Q = \ln \sqrt{k}$ . We shall call one of the asymptotic families of the surface the first; we shall use the usual conditions of orientation of positive directions on the asymptotics and of the positive side of the surface. Let the angle between the positive directions of the asymptotics be denoted by  $\omega$ , and the natural parameters (arcs) of the asymptotics by  $s_1, s_2$ . Suppose that on some segment  $OP$  of some asymptotic of the first family the angle  $\omega$  varies monotonically (the direction from  $O$  to  $P$  is assumed positive). Through all points of  $OP$  draw the asymptotics of the second family, and on each of them lay off a segment of length  $B$ , in the positive direction if  $\omega'_{s_1} > 0$  on  $OP$ . The region filled by these segments will be denoted by  $\Omega$ ; the segment  $OP$  will be called the base of  $\Omega$ , and the segments of the second family passing through  $O$  and  $P$  will be called, respectively, the left and right sides. Suppose that along the base

$$\frac{1}{k_0^2}(\omega'_{s_1})_0 \geq m_0 > 0, \quad (m_0 = \text{const}, k_0 = k(M), (\omega'_{s_1})_0 = \omega'_{s_1}(M), M \in OP).$$

Let the length of the base be equal to  $a_0 \leq 1$ .

Put

$$\frac{1}{\Delta} = \max \left\{ \sup \left| \frac{k'_s}{k} \right|; \sup \left| \frac{k''_{ss}}{k} \right|_g \right\},$$

where  $k'_s$  is the derivative in any direction,  $(k''_{ss})_g$  is the second derivative along the arc of any geodesic; the upper bounds are taken over the region  $\Omega$ .

**Theorem.** *If*

$$\frac{1}{\Delta} \leq \frac{1}{6}, \quad |\text{grad } \omega| \leq \frac{\Delta}{12}$$

*throughout the entire region  $\Omega$ , then  $B$  is bounded by a certain quantity depending only on  $a_0$  and  $m_0$ . For example,*

$$B \leq \frac{768\pi}{a_0 \sin \frac{m_0 a_0}{12}}.$$

We shall give the main stages of the proof. Denote by  $u, v$  local asymptotic coordinates, assuming that along the asymptotics of the first family  $v = \text{const}$ . From the general formulas of the theory of surfaces, by means of rather lengthy computations, one obtains the equation

$$k^2 \frac{\partial}{\partial s_2} \left( \frac{1}{k^2} \frac{\partial \omega}{\partial s_1} \right) = \{k^2 + A + \alpha |\text{grad } \omega|\} \sin \omega, \quad (1)$$

where

$$A = (Q''_{s_1 s_1})_g + (Q''_{s_2 s_2})_g - 4Q'_{s_1} Q'_{s_2} \cos \omega,$$

$$\alpha = Q'_{s_2} \sin \psi_1 - 2Q'_{s_1*} \cos \psi_1 - 2Q'_{s_2} \sin \psi_2;$$

$$\frac{\partial}{\partial s_1} = \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial s_2} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial v};$$

$\psi_1, \psi_2$  are the angles of the first and second asymptotics with the direction of the gradient of  $\omega$ ;  $Q'_{s_1*}$  is the derivative in the direction orthogonal to the first asymptotic.

According to the assumptions,

$$\left| \frac{A}{k^2} \right| \leq \frac{3}{\Delta} \leq \frac{1}{2}, \quad \left| \frac{\alpha}{k^2} \right| \leq \frac{3}{\Delta}.$$

Hence, and from equation (1),

$$\frac{\Delta}{12} \geq |\text{grad } \omega| \geq \frac{\partial \omega}{\partial s_1} \geq \frac{1}{k^2} \frac{\partial \omega}{\partial s_1} - \frac{1}{k_0^2} \left( \frac{\partial \omega}{\partial s_1} \right)_0 \geq \frac{1}{4} \int_0^{s_2} \sin \omega \, ds_2 > 0. \quad (2)$$

Thus,

$$\int_0^{s_2} \sin \omega \, ds_2 \leq \frac{\Delta}{3}. \quad (3)$$

On the other hand, the following relation holds:

$$\frac{\partial \ln(k\sqrt{E})}{\partial s_2} = \sin \omega \frac{\partial Q}{\partial s_1^*}. \quad (4)$$

If along the base  $k_0\sqrt{E_0} = 1$ , then from (3) and (4) we obtain, throughout the entire region,

$$\frac{1}{2} < e^{-1/6} \leq \sqrt{E} \leq e^{+1/6} < 2. \quad (5)$$

Let  $a$  be the length of an arbitrary segment  $u_1 \leq u \leq u_2$  of the base; it follows from (5) that the length of every asymptotic segment of the first family lying in the region  $\Omega$  between the asymptotics  $u = u_1$ ,  $u = u_2$  is equal to  $\lambda a$ , where  $1/4 < \lambda < 4$ . Similarly, one can show that if  $b$  is the length of any segment  $v_1 \leq v \leq v_2$  of the left side of  $\Omega$ , then  $\mu b$  is the length of an arbitrary asymptotic segment of the second family lying in the region  $\Omega$  between the asymptotics  $v = v_1$ ,  $v = v_2$ , where  $1/4 < \mu < 4$ .

Denote by  $b_1$  the segment of the left side of the region  $\Omega$  which begins at the point  $O$  and has length equal to  $\frac{1}{4}B$ .

It follows from the estimates for  $\mu$  that every asymptotic of the first family entering the region  $\Omega$  from some point of the segment  $b_1$  reaches the right side of the region  $\Omega$  while remaining inside  $\Omega$ ; moreover, the asymptotic issuing from the end of  $b_1$  cuts off, together with the base of the region  $\Omega$ , segments on the asymptotics of the second family of length  $\geq \frac{1}{16}B$ .

Let  $\delta$  be a segment of an asymptotic of the first family which begins somewhere on  $b_1$  and intersects  $\Omega$ . Divide the base of the region  $\Omega$  into three equal parts and draw through the points of division inside  $\Omega$  asymptotics of the second family; they divide  $\delta$  into three parts:  $\delta_1, \delta_2, \delta_3$  (counting the indices from the left side of  $\Omega$ ). By the estimates for  $\lambda$ , the length  $\delta_i > \frac{1}{12}a_0$  ( $i = 1, 2, 3$ ); moreover, from (2) we have  $\partial\omega/\partial s_1 \geq m_0$  on the whole segment  $\delta$ . Therefore, on the segment  $\delta_2$  one has  $\sin \omega \geq \sin \frac{m_0 a_0}{12}$ . Hence, again from (2), for the segment  $\delta$  issuing from the end of  $b_1$ , we obtain on  $\delta_2$

$$\frac{\partial\omega}{\partial s_1} \geq \frac{1}{4} \int_0^{s_2} \sin \omega ds_2 \geq \frac{1}{64} B \sin \frac{m_0 a_0}{12}. \quad (6)$$

Since  $0 < \omega < \pi$ , from (6) and from the inequality  $\delta_2 > a_0/12$  we have

$$Ba_0 \sin \frac{m_0 a_0}{12} < 768\pi,$$

which is what the theorem asserts.

item 6°. There is another generalization of the strengthened Hilbert theorem, which we shall state only in general terms. Namely: for an asymptotic  $L$  there cannot be an  $\varepsilon$ -strip on which the quantities  $|k_s/k|$ ,  $|k_{ss}/k|_g$  are sufficiently small, if the integral  $\int_L \sin \omega ds$  is sufficiently large (in comparison with the upper bounds of these quantities;  $\varepsilon$  also depends on them).

item 7°. The author found equation (1) relying on certain data from joint work with E. G. Poznyak. The derivation of equation (1), in turn, prompted further joint work, in the course of which other convenient transformations of the basic equations of the theory of surfaces were established, as well as a precise formulation of the assertion of item 6°, its complete proof, and substantial consequences. These results will be reported in joint publications with E. G. Poznyak.

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*Note: Figure translations are in progress. See original paper for figures.*

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