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V. I. ZHEGALOV

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**Abstract**

**Full Text**

**Reports of the Academy of Sciences of the USSR**

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**MATHEMATICS**

**V. I. ZHEGALOV**

**A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE OF HIGHER ORDER**

*(Presented by Academician I. N. Vekua, July 29, 1960)*

1. Let  $D$  be a simply connected domain of the plane  $z = x + iy$ , bounded by a Jordan line  $\sigma$  with endpoints at the points  $A(0, 0)$  and  $B(1, 0)$ , lying in the upper half-plane  $y > 0$ , and by the characteristics  $AC : x + y = 0$  and  $CB : x - y = 1$  of the equation

$$\left( \frac{\partial^2}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2}{\partial y^2} \right)^n u = 0. \tag{1}$$

Obviously, the  $x$ -axis is the line of degeneration of the type of equation (1).

The following problem is considered:

Determine the function  $u(x, y)$ , which is a solution of equation (1) in the domain  $D$  for  $y \neq 0$ , continuous up to the boundary, and possessing continuous partial derivatives up to order  $(2n - 1)$  inclusive everywhere in the domain  $D$ , with the possible exception of neighborhoods of the points  $A$  and  $B$ , where derivatives of order  $(2n - 1)$  may become infinite of order less than unity, if on the line  $\sigma$  and on one of the characteristics, for example on  $AC$ , the conditions

$$\left( \frac{\partial^2}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2}{\partial y^2} \right)^k u = \begin{cases} \varphi_k(\tau) & \text{on } \sigma, \\ \psi_k(x) & \text{on } AC, \end{cases} \tag{2}$$

$$\varphi_k(0) = \psi_k(0), \quad k = 0, 1, \dots, n - 1.$$

Here  $\varphi_k, \psi_k$  are prescribed functions, with  $\psi_k$  continuously differentiable  $2n - 2k$  times, and  $\varphi_k$  continuously differentiable  $2n - 2k - 1$  times. Without loss of generality, one may assume

$$\varphi_k(0) = \varphi_k(1) = 0.$$

The importance of considering problems of this kind was emphasized by A. V. Bitsadze <sup>(3)</sup>. The first attempt at solving a problem analogous to the one formulated, for  $n = 2$ , was made by R. Ya. Agishev <sup>(1)</sup>. In the present note a proof is proposed of the existence and uniqueness of the solution of the formulated problem.

2. In solving the problem, certain relations from the theory of polyanalytic functions are used. It is known <sup>(4)</sup> that an  $n$ -harmonic function  $u(x, y)$  can be represented in the form

$$u(x, y) = \operatorname{Re} \sum_{s=0}^{n-1} \bar{z}^s \chi_s(z), \quad (3)$$

where the analytic functions  $\chi_s(z)$  are uniquely determined by  $u(x, y)$ , if, for example, the conditions

$$\chi_s^{(s)}(0) = \overline{\chi_s^{(s)}(0)} \quad (s = 0, \dots, n-1);$$

$$\chi_s^{(m)}(0) = 0 \quad (m = 0, \dots, s-1; s = 1, \dots, n-1)$$

are satisfied.

The function

$$v(x, y) = \operatorname{Im} \sum_{s=0}^{n-1} \bar{z}^s \chi_s(z)$$

is called the polyharmonic function conjugate to  $u(x, y)$ , and the expression

$$F(z, \bar{z}) = u(x, y) + iv(x, y) = \sum_{s=0}^{n-1} \bar{z}^s \chi_s(z)$$

is called a polyanalytic function of order  $n$ .

It follows from what has been said that  $v(x, y)$ , and hence also  $F(z, \bar{z})$ , are determined by  $u(x, y)$  up to  $n^2$  real constants.

By the method of induction one can prove that the functions  $u(x, y)$  and  $v(x, y)$  thus defined satisfy the relations

$$\sum_{s=0}^n (-1)^s C_n^{2s} \frac{\partial^n u(x, y)}{\partial x^{n-2s} \partial y^{2s}} = \sum_{s=0}^n (-1)^s C_n^{2s+1} \frac{\partial^n v(x, y)}{\partial x^{n-2s-1} \partial y^{2s+1}}, \quad (4)$$

$$\sum_{s=0}^n (-1)^s C_n^{2s} \frac{\partial^n v(x, y)}{\partial x^{n-2s} \partial y^{2s}} = - \sum_{s=0}^n (-1)^s C_n^{2s+1} \frac{\partial^n u(x, y)}{\partial x^{n-2s-1} \partial y^{2s+1}}. \quad (5)$$

3. We shall seek the solution of the problem for the case when  $\sigma$  is the semicircle:  $|z - \frac{1}{2}| = \frac{1}{2}$ ,  $y > 0$ . The proof of the existence of a solution for the case when  $\sigma$  is any smooth Jordan curve can be obtained by means of a conformal mapping.

Let  $D_1$  and  $D_2$  denote those parts of the domain  $D$  in which, respectively,  $y > 0$  and  $y < 0$ .

Replacing equation (1) by the equivalent system

$$\frac{\partial^2 u}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2 u}{\partial y^2} = u_1(x, y), \quad (6)$$

$$\frac{\partial^2 u_r}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2 u_r}{\partial y^2} = u_{r+1}(x, y) \quad (r = 1, \dots, n-1) \quad (7)$$

and taking into account the boundary conditions (2), we obtain, for equation (6) and for each of the equations of the system (7), the problem  $T$  (we use the terminology of A. V. Bitsadze<sup>(2)</sup>). We solve successively the resulting problems, beginning with the last one. Since  $u_n = 0$ , the last equation of the system (7) is the M. A. Lavrent'ev equation. Consequently, the function  $u_{n-1}(x, y)$  is found from the solution of problem  $T$  for this equation<sup>(2)</sup>. All the remaining problems are of the same character: for each of them the differential equation is inhomogeneous, but with a known right-hand side. In order to explain the scheme for solving these problems, we dwell in more detail on the first of them.

Taking into account that on  $AC$ ,  $u = \psi_0(x)$ , we easily obtain the general representation of the solution of equation (6) in the domain  $D_2$ . It has the form

$$u(x, y) = G(x, y) + \Phi_1(x + y), \quad (8)$$

where  $G(x, y)$  is known, and  $\Phi_1$  is an arbitrary  $2n$ -times differentiable function.

In the domain  $D_1$ ,  $u(x, y)$  is an  $n$ -harmonic function. Introduce the conjugate  $n$ -harmonic function  $v(x, y)$  satisfying the conditions

$$\frac{\partial^k \Delta^m v(0, 0)}{\partial x^k} = 0 \quad (k = 0, \dots, n - m - 1; m = 0, \dots, n - 1). \quad (9)$$

Using the representation (8), as well as the continuity property of the partial derivatives of the function  $u(x, y)$ , from conditions (5) we obtain

$$-a_k \Delta^{n-k} v(x, 0) = 2^{n-k} \Phi_1^{(2n-2k)}(x) + Q_k(x) \quad (k = 1, \dots, n). \quad (10)$$

Here the functions  $Q_k(x)$  and the constants  $a_k$ , by virtue of conditions (9), are completely determined.

If, using formula (8), we compute  $\Delta^{n-k} u(x, 0)$  ( $k = 1, \dots, n$ ), then, taking into account relations (10), we obtain on  $AB$  the conditions

$$\Delta^{n-k} u(x, 0) + a_k \Delta^{n-k} v(x, 0) = P_k(x) \quad (k = 1, \dots, n), \quad (11)$$

where  $P_k(x)$  are completely determined functions.

Thus we arrive at the following problem:

Find in the domain  $D_1$  a polyanalytic function  $F(z, \bar{z}) = u(x, y) + iv(x, y)$ , if on the boundary of  $D_1$  the conditions

$$\Delta^{n-k} u(\tau) + b_k(\tau) \Delta^{n-k} v(\tau) = c_k(\tau), \quad (12)$$

$$b_k(\tau) = \begin{cases} 0, & \tau \in \sigma, \\ a_k, & \tau \in AB, \end{cases} \quad c_k(\tau) = \begin{cases} \varphi_{n-k}(\tau), & \tau \in \sigma, \\ P_k(\tau), & \tau \in AB \end{cases} \quad (k = 1, \dots, n).$$

As M. P. Ganin showed <sup>5</sup>, this problem is equivalent to  $n$  Hilbert problems for the indicated semicircle, whose solution in explicit form can be obtained by using the method of L. I. Chibrikova <sup>6</sup>.

The real part of the resulting polyanalytic function gives the solution of the problem formulated by us in the domain  $D_1$ . If it is denoted by  $u^*(x, y)$ , then in the domain  $D_2$  the solution is given by the formula

$$u(x, y) = u^*(x + y, 0) + G(x, y) - G(x + y, 0).$$

The uniqueness of the solution follows from the fact that, under zero boundary conditions, all equations of system (7) and equation (6) are transformed into equations of M. A. Lavrent'ev, for each of which problem T will have only the zero solution.

In conclusion, the author expresses deep gratitude to L. I. Chibrikova for guidance of the work.

Kazan State University  
named after V. I. Ulyanov-Lenin

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*Note: Figure translations are in progress. See original paper for figures.*

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