R. M. Trigub

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Abstract

Full Text

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APPROXIMATION OF FUNCTIONS BY POLYNOMIALS WITH INTEGER COEFFICIENTS

(Presented by Academician S. N. Bernstein, 15 V 1961)

The article considers certain questions in the theory of approximation of functions by polynomials

$$Q_n(x) = \sum_{k=0}^{n} c_k x^k \tag{1}$$

with integer coefficients on an interval of the real axis. In contrast to approximation by polynomials with arbitrary coefficients, the possibility of approximation by polynomials (1) depends on certain arithmetical properties of the interval and of the function being approximated. If some function, different from a polynomial, is the limit of a sequence of polynomials with integer coefficients converging uniformly on an interval $[a, b]$, then there exists a polynomial $X(x) \not\equiv 0$ with integer coefficients such that $\max_{[a,b]} |X(x)| < 1$. Therefore such approximation is impossible already on an interval of length 4. On the other hand, a continuous function admitting the indicated approximation on an interval containing at least one integral point must satisfy certain arithmetical conditions, the number of which grows without bound as the length of the interval approaches 4.

A Weierstrass-type theorem for any interval of length less than 4 was proved by Okada ([7])*. For the interval $[0, 1]$ this theorem was sharpened in the works ([2],[6],[4]).

In the present work direct theorems are obtained for approximation by polynomials (1) on any interval of length less than 4. We state these results.

1. We begin with an interval containing no integral points.

Theorem 1. If the function $f(x)$ is continuous on the interval $[\delta, 1-\delta]$ $(0 < \delta < 1/2)$, then for any n

$$E_n^e(f; [\delta, 1-\delta]) = \inf_{Q_n} \max_{[\delta,1-\delta]} |f(x) - Q(x)| \le E_n + 2n\rho^n,$$

where

$$\rho = \max\left\{\frac{1}{2}, \frac{1-2\delta}{1+2\sqrt{\delta(1-\delta)}}\right\},$$

and E_n is the best approximation by arbitrary polynomials. This theorem strengthens one result of Kuz' min-Kantorovich (^{6}).

Corollary. If λ is a nonintegral real number, and $\delta \leq 1/10$, then

$$\lim_{n\to\infty} \sqrt[n]{E_n^e(\lambda; [\delta, 1-\delta])} = \frac{1-2\delta}{1+2\sqrt{\delta(1-\delta)}}.$$

If, however, $\delta > 1/10$ and λ is not a dyadic-rational number, then

$$\lim_{n\to\infty} \sqrt[n]{E_n^e(\lambda; [\delta, 1-\delta])} = \frac{1}{2}.$$

Thus an answer has been obtained to one question of S. N. Bernstein (^{2}).

* The existence of polynomials $X(x)$ on $[a, b]$ $(b-a<4)$ was proved by Fekete (^{9}).

2. The following theorem of Jackson type is valid:

Theorem 2. If a function $f(x)$ has on the interval $[a, b]$ $(b-a<4)$ an r-th (r an integer ≥ 0) continuous derivative with modulus of continuity $\omega(h)$, and its derivatives $f^{(\nu)}(x)$ $(\nu = 0, \dots, r)$ vanish at all zeros of a certain polynomial $X(x)$ $(\max_{a\leq x\leq b} |X(x)| < 1)$ lying on $[a, b]$, then for every n there exists a polynomial $Q_n(x)$ with integer coefficients such that, for $x \in [a, b]$ and $\nu = 0, \dots, r$,

$$\left|f^{(\nu)}(x) - Q_n^{(\nu)}(x)\right| \leq C_r \frac{\omega(1/n)}{n^{r-\nu}},$$

where C_r does not depend on x or n.

For the interval $[0, 1]$ this theorem was proved by A. O. Gelfond (^{4}).

Under the arithmetic conditions imposed on the function in Theorem 2, approximation by polynomials (1) cannot be, generally speaking, best in order for arbitrary differential properties of the function. If one abandons simultaneous approximation of the function and its derivatives, then in some cases (when $X(a)\cdot X(b) = 0$) the number of arithmetic conditions can be reduced. We state the corresponding theorem only for the interval $[-1, 1]$.

Theorem 2′. Let the function $f(x)$ have on $[-1, 1]$ an r-th continuous derivative with modulus of continuity $\omega(h)$. In order that, for some C_r and every n, the inequality

$$E_n^i(f; [-1, 1]) \leq C_r \frac{\omega(1/n)}{n^r}$$

hold, it is sufficient (when $\omega(h) = h^\alpha$, $0 < \alpha \leq 1$, also necessary) that the polynomial $q(x)$, defined by the conditions

$$q^{(\nu)}(0) = f^{(\nu)}(0) \qquad (\nu = 0, \dots, r),$$

$$q^{(\nu)}(1) = f^{(\nu)}(1) \qquad \left(\nu = 0, \dots, \left[\frac{r}{2}\right]\right),$$

$$q^{(\nu)}(-1) = f^{(\nu)}(-1) \qquad \left(\nu = 0, \dots, \left[\frac{r}{2}\right]\right),$$

be a polynomial with integer coefficients.

3. Theorem 2 admits a strengthening.

Theorem 3. Under the conditions of Theorem 2 there exists a polynomial $Q_n(x)$ such that, for all $x \in [a, b]$ and $\nu = 0, \dots, r$,

$$\left|f^{(\nu)}(x) - Q_n^{(\nu)}(x)\right| \leq C_r \left(\frac{\sqrt{(x-a)(b-x)}}{n} + \frac{1}{n^2}\right)^{r-\nu} \omega\left(\frac{\sqrt{(x-a)(b-x)}}{n} + \frac{1}{n^2}\right).$$

For approximation by polynomials with arbitrary coefficients and $\nu = 0$, this theorem was obtained by A. F. Timan (^{8}).

Theorem 4. If a function $f(x)$ has on $[a, b]$ $(b - a < 4)$ an r-th derivative satisfying Zygmund' s condition*, and its derivatives $f^{(\nu)}(x)$ $(\nu = 0, \dots, r)$ vanish at all zeros of $X(x)$ lying on $[a, b]$, then for every n there is a polynomial $Q_n(x)$ such that, for all $x \in [a, b]$ and $\nu = 0, \dots, r$,

$$\left|f^{(\nu)}(x) - Q_n^{(\nu)}(x)\right| \leq C_r \left(\frac{\sqrt{(x-a)(b-x)}}{n} + \frac{1}{n^2}\right)^{r+1-\nu}.$$

For approximation by polynomials with arbitrary coefficients this theorem was proved independently in (^{6}) and (^{10}).

* $\left|f^{(r)}(x-h) - 2f^{(r)}(x) + f^{(r)}(x+h)\right| \leq h, \quad x \pm h \in [a, b].$

4. Let us dwell further on approximation in the integral metric.

Theorem 5. If $f(x) \in L_p$ $(1 \leq p < \infty)$ on $[a, b]$ $(b - a < 4)$, then

$$E_n^e(f; [a, b])_{L_p} = \inf_{Q_n} \left(\int_a^b |f(x) - Q_n(x)|^p \, dx\right)^{1/p} =$$

$$= E_n(f; [a, b])_{L_p} + O\left(\frac{1}{n^{1/2}p}\right). \qquad (2)$$

This theorem strengthens the theorem of E. Aparicio ((1), see also (4)). In the case of the interval $[0,1]$, instead of $O\left(\frac{1}{n^{1/2}p}\right)$ one may put $O\left(\frac{1}{n^{1/p}}\right)$. The latter estimate, as well as estimate (2), for an interval containing at least one integer point in its interior, cannot be improved, at least for $p = 2$.

5. Theorems 2 and 5 are also valid for approximation on the set $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ $(0 < \alpha < \beta)$, if $\beta^2 - \alpha^2 < 4$.

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Dnepropetrovsk State University
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Note: Figure translations are in progress. See original paper for figures.