ON THE BEHAVIOR OF CORRECTIONS IN SOME METHODS FOR APPROXIMATE COMPUTATION OF THE BOUNDS OF THE SPECTRUM OF A SELF-ADJOINT OPERATOR

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Abstract

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ON THE BEHAVIOR OF CORRECTIONS IN SOME METHODS FOR APPROXIMATE COMPUTATION OF THE BOUNDS OF THE SPECTRUM OF A SELF-ADJOINT OPERATOR

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1. Let A be a linear bounded positive definite self-adjoint operator, and suppose that it is required to find one of the bounds m of its spectrum. Usually, for this purpose, a certain sequence of elements $\{x_n\}$ is constructed such that the numerical sequence

$$\mu_n = \frac{(Ax_n, x_n)}{(x_n, x_n)} \qquad (n = 0, 1, 2, ...) \tag{1}$$

converges to the desired bound of the spectrum. In constructing the sequence $\{x_n\}$, one has to compute correction vectors $\delta_n = x_{n+1} - x_n$. At the seminar on functional analysis of Voronezh State University, M. A. Krasnosel' skii put forward the supposition (see also ([1])) that in a number of methods the correction vectors δ_n can be used to obtain additional information about the spectrum of the operator A. It turned out that this is indeed so. Some results in this direction are indicated in ([2,5]). Here new results are presented.

2. Denote the upper and lower bounds of the spectrum of the operator A, respectively, by M and m $(0 < m \leq M)$. A sequence of elements $\{x_k\}$ of the space H will be called **extremal** for the functional

$$\mu(x) = \frac{(Ax, x)}{(x, x)}, \tag{2}$$

if

$$\lim_{k\to\infty} \mu(x_k) = m \tag{3}$$

or

$$\lim_{k\to\infty} \mu(x_k) = M. \tag{4}$$

In this case, if equality (3) is fulfilled, then the sequence $\{x_k\}$ will be called **minimizing**, and if equality (4) is fulfilled, **maximizing**.

Let E_λ denote the spectral function of the operator A; let $\Delta(x)$ denote the projection of the element Ax onto the subspace orthogonal to the element x. It is easy to verify that

$$\Delta(x) = Ax - \mu(x)x. \tag{5}$$

A maximizing sequence of elements $\{x_k\}$ will be called **biextremal for the functional** $\mu(x)$, if for the corresponding sequence $\{\Delta_k = \Delta(x_k)\}$ the relations

$$\|\Delta_k\| \neq 0 \quad (k = 0, 1, 2, ...), \qquad \lim_{k\to\infty} \mu(\Delta_k) = m.$$

We shall call a minimizing sequence $\{x_k\}$ **strongly minimizing** if the following conditions are satisfied: a) m is an isolated point of the spectrum of the operator A; b) $\|\Delta_k\| \neq 0$ $(k = 0, 1, 2, ...)$, $\lim_{k\to\infty} \mu(\Delta_k) = m_1$, where m_1 is the smallest number of the spectrum greater than m.

Analogously, we shall call a maximizing sequence of elements **strongly maximizing** if: a) M is an isolated point of the spectrum of the operator A; b) $\|\Delta_k\| \neq 0$ $(k = 0, 1, 2, ...)$, $\lim_{k\to\infty} \mu(\Delta_k) = M_1$, where M_1 is the greatest number of the spectrum less than M.

In what follows it is useful to keep in mind the following simple facts:

1) If m (respectively M) is not an eigenvalue of the operator A, then a minimizing (respectively maximizing) sequence of normalized elements $\{\bar{x}_k = x_k/\|x_k\|\}$ converges weakly to zero in the space H.

2) If m (respectively M) is an isolated eigenvalue of the operator A of finite multiplicity, then a minimizing (respectively maximizing) sequence of normalized elements $\{\bar{x}_n = x_n/\|x_n\|\}$ converges "in direction" to some vector e corresponding to the eigenvalue m (respectively M), i.e.

$$\lim_{n\to\infty} \sin^2 \widehat{(\bar{x}_n, e)} = \lim_{n\to\infty} \left[1 - (\bar{x}_n, e)^2\right] = 0.$$

3) If m_1 (respectively M_1) is not an eigenvalue of the operator A and the sequence $\{x_k\}$ is strongly minimizing (respectively strongly maximizing) for the functional $\mu(x)$, then the corresponding sequence $\{\Delta_k = \Delta(x_n)/\|\Delta(x_k)\|\}$ converges weakly to zero in the space H.

4) If, however, m_1 (respectively M_1) is an isolated eigenvalue of the operator A of finite multiplicity and the sequence $\{x_k\}$ is strongly minimizing (respectively strongly maximizing) for the functional $\mu(x)$, then the sequence $\{\bar{\Delta}_k\}$ converges "in direction" to some eigenvector e_1 corresponding to the number m_1 (respectively M_1).

Below two methods of approximate computation of the bounds of the spectrum of the operator A are considered, and it is established that the sequence of approximations $\{x_k\}$ obtained in their realization, under certain conditions, is either strongly minimizing, or extremal, or strongly maximizing.

3. To find the lower bound m of the spectrum of the operator A, M. A. Krasnosel' skii proposed ([1]) an iterative process in which the sequence of approximations is defined by the equalities

$$x_{k+1} = x_k - \frac{1}{\gamma_k}\Delta_k, \tag{6}$$

where

$$\Delta_k = Ax_k - \mu_k x_k, \qquad \mu_k = \mu(x_k), \qquad \gamma_k = \mu(\Delta_k). \tag{7}$$

B. P. Pugachev showed that the sequence $\{x_k\}$ obtained by formulas (6)–(7) is minimizing for the functional $\mu(x)$, and obtained estimates of the rate of convergence of the process under consideration. It turns out that, under certain restrictions on the spectrum of the operator A, one can assert that this sequence is strongly minimizing for the functional $\mu(x)$. Namely, the following holds:

Theorem 1. Suppose:

1) m is an isolated point of the spectrum of the operator A, and

$$m > \frac{M - m_1}{2};$$

2) the initial approximation x_0 is chosen so that, for any $\varepsilon > 0$,

$$\|E_{m+\varepsilon}x_0\| > 0, \qquad \|(E_{m_1+\varepsilon} - E_{m_1-\varepsilon})x_0\| > 0.$$

Then the sequence of approximations $\{x_k\}$, obtained from the element x_0 by means of the process (6)–(7), is strongly minimizing for the functional $\mu(x)$.

Thus, the iterative process (6)–(7) can be used, generally speaking, for the simultaneous computation of two points of the spectrum of the operator A, which under certain conditions are its eigenvalues. In the latter case the corresponding eigenvectors can also be obtained with its aid.

4. V. N. Kostarchuk investigated in [3] an iterative process for the approximate computation of the upper bound M of the spectrum of the operator A, in which the sequence of approximations is constructed by the formula

$$x_{k+1} = x_k - \frac{2}{\mu_k} A x_k. \tag{8}$$

The latter can evidently also be written as

$$x_{k+1} = -\left(x_k + \frac{2}{\mu_k}\Delta_k\right). \tag{9}$$

With respect to the process (9) the following holds:

Theorem 2. Suppose:

1) M is an isolated point of the spectrum of the operator A, and $M < 2m$;
2) the initial approximation x_0 is chosen so that, for any $\varepsilon > 0$,

$$\|(E - E_{M-\varepsilon})x_0\| > 0, \qquad \|(E_{M_1+\varepsilon} - E_{M_1-\varepsilon})x_0\| > 0.$$

Then the sequence of approximations $\{x_k\}$, obtained from the element x_0 by means of the process (9), is strongly maximizing for the functional $\mu(x)$.

Let $L(m, \varepsilon)$ be the subspace of the space H onto which the operator $E_{m+\varepsilon}$ projects; $H(m, \varepsilon)$ the orthogonal complement to $L(m, \varepsilon)$ in the space H. We shall say that the iterative process (9) **degenerates into the subspace** $H(m, \varepsilon)$ if the sequence of approximations $\{x_k\}$ has a pair of elements x_{k_0}, x_{k_0+1} such that $x_{k_0} \in H(m, \varepsilon)$, $x_{k_0+1} \in H(m, \varepsilon)$. By G_ε we shall denote the set of all elements of the space H possessing the property that if the initial approximation x_0 is chosen in G_ε, then the process (9) degenerates into the subspace $H(m, \varepsilon)$.

Theorem 3. Suppose:

1) M is an isolated point of the spectrum of the operator A, and $M > 2M_1$;
2) the initial approximation x_0 is chosen so that, for any $\varepsilon > 0$,

$$\|(E - E_{M-\varepsilon})x_0\| > 0, \qquad \|E_{m+\varepsilon}x_0\| > 0,$$

and, for $\varepsilon < \varepsilon_0$,

$$x_0 \notin G_\varepsilon,$$

where ε_0 is some fixed number.

Then the sequence of approximations $\{x_k\}$, obtained from x_0 by means of the process (9), is biextremal for the functional $\mu(x)$.

We note that $x_0 \notin G_\varepsilon$ for $\varepsilon < \varepsilon_0$, if

$$\mu(x_0) = \frac{(Ax_0, x_0)}{(x_0, x_0)} > 2(m + \varepsilon_0).$$

Theorems 2 and 3 are also valid for the so-called α-processes considered by V. P. Kostarchuk in [(3)] (by analogy with the α-processes for solving systems of linear algebraic equations proposed by M. A. Krasnosel' skii and S. G. Krein [(4)]).

5. In conclusion, we note that the proof of Theorems 1-3 and of analogous theorems previously published by the author in [(5)] can be obtained by means of a certain general scheme, which we do not present here because of its unwieldiness.

By arguments analogous to those given in [(5)], one can (with the aid of Theorems 1-3) establish formulas for accelerating the convergence of the methods considered above.

It should also be noted that, by passing from the operator A to the operator $A_1 = A - kI$ (where k is a certain number and I is the identity operator), one can, generally speaking, always arrange that the conditions of one of the formulated theorems be satisfied.

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REFERENCES

[1] M. A. Krasnosel' skii, UMN, **11**, no. 6, 151 (1956).
[2] B. P. Pugachev, Tr. Seminar on Functional Analysis, Voronezh State Univ., no. 3-4 (1956).
[3] V. N. Kostarchuk, ibid.
[4] M. A. Krasnosel' skii, S. G. Krein, Matem. sborn., **31**, no. 2, 315 (1952).
[5] G. A. Bessmertnykh, DAN, **128**, no. 6, 1106 (1959).

Note: Figure translations are in progress. See original paper for figures.